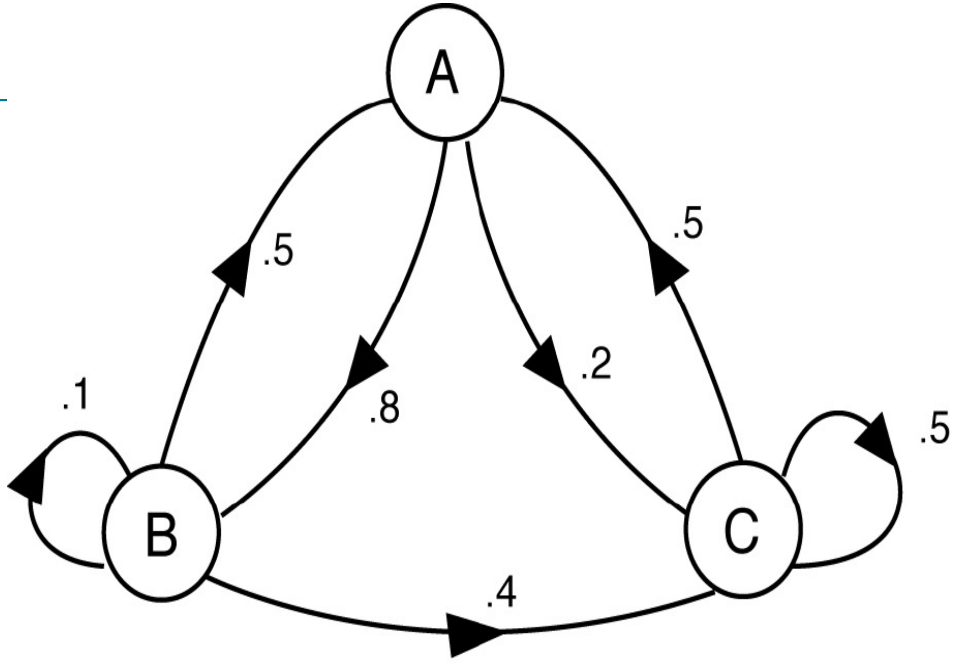
Math 22 –
Linear Algebra and its applications

Instructor: Bjoern Muetzel

Applications

NETWORKS, MARKOV CHAINS AND

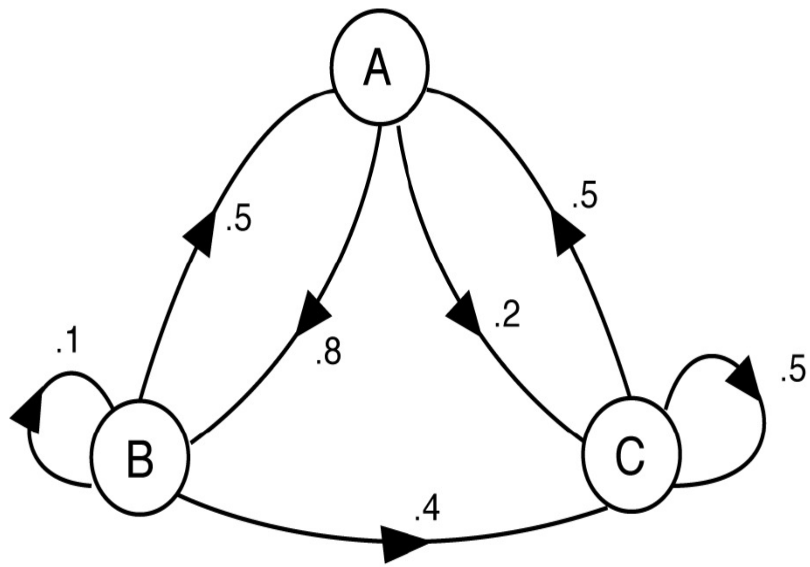
GOOGLE'S PAGE RANK ALGORITHM



Markov graph of transiton probabilites between states A, B and C

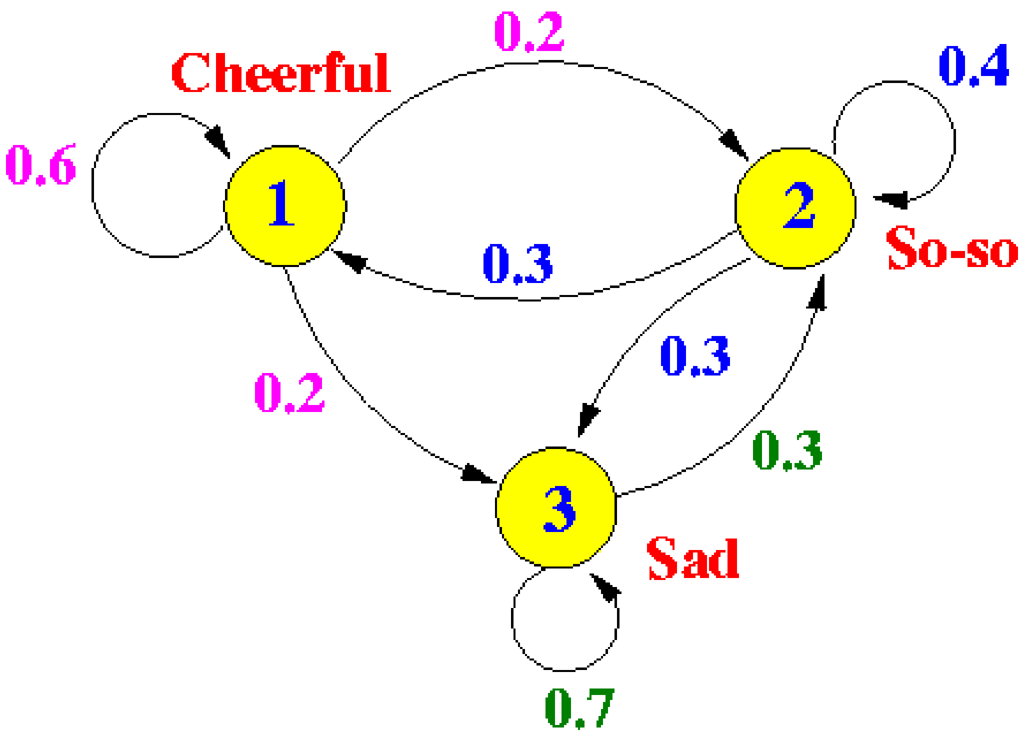
<u>Summary:</u>

Transitions or **flows** in **networks** can be analyzed by writing the information into a **matrix**. Finding the **steady state** of the system amounts to finding an **eigenvector of this matrix**.



Markov graph of transiton probabilites
between states A, B and C

EXAMPLE: IN THE MOOD



EXAMPLE: IN THE MOOD

- **Exercise:** Draw your own (imaginary) mood network with transition probabilities.

MARKOV CHAINS

- **Definition: 1.)** A **probability vector $\boldsymbol{v}$** in $\mathbb{R}^n$ is a vector with non-negative entries (probabilities) that add up to 1.

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \quad v_1 + v_2 + \cdots + v_n = 1, \quad \text{especially} \quad v_i \text{ in } [0,1].$$

2.) A **stochastic matrix $\boldsymbol{P}$** is an $n \times n$ matrix whose columns are probability vectors.

3.) A **Markov chain** is a **sequence of probability vectors $(\boldsymbol{x_k})_{\boldsymbol{k} \textbf{ in } \mathbb{N}}$**, together with a stochastic matrix P, such that $\boldsymbol{x_0}$ **is the initial state** and

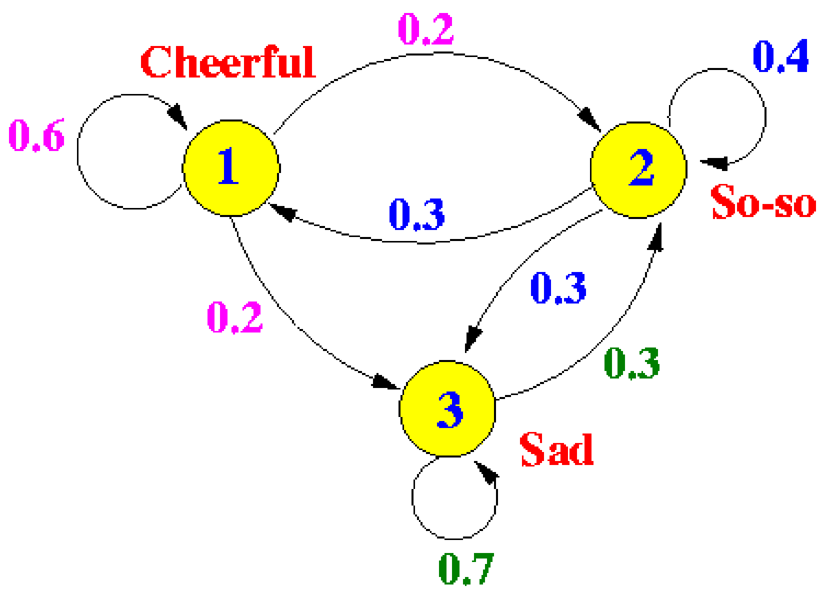
$$\boldsymbol{x_k = P^k x_0} \quad \text{or equivalently} \quad \boldsymbol{x_k = P x_{k-1}} \quad \text{for all } \boldsymbol{k} \textbf{ in } \mathbb{N}\setminus\{\boldsymbol{0}\}.$$

4.) A vector $\boldsymbol{x_k}$ of a Markov chain is called a **state vector.**

Visualization: We can visualize Markov chains with directed graphs.

- **Vertices** of the graph represent the **states** (entries of a vector).
- **Arrows** of the graph represent the **transitions** and their probability.
- We can write the transition probabilities into a stochastic matrix *P*.

Example:



$$\begin{array}{cc} & \begin{array}{ccc} \textbf{From: } 1 & 2 & 3 \end{array} & \textbf{To:} \\ P= & \begin{bmatrix} 0.6 & 0.3 & 0 \\ 0.2 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.7 \end{bmatrix} & \begin{array}{c} \textbf{1} \\ \textbf{2} \\ \textbf{3} \end{array} \end{array}$$

MARKOV CHAINS

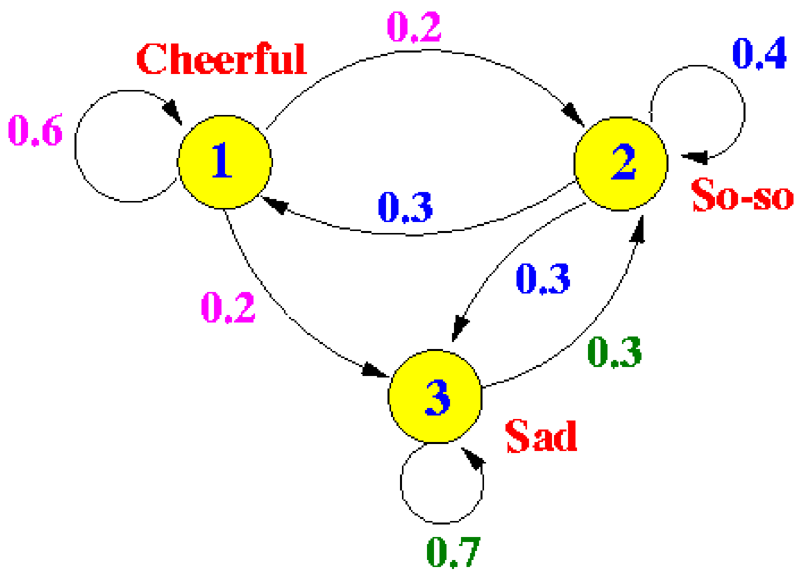
Visualization:

- The index **k** of a state vector $\boldsymbol{x_k}$ represents the **time interval**.
  $\boldsymbol{x_k}$ describes the state of the system at the time interval **k**.
- If $\boldsymbol{x_k} = (\boldsymbol{x_{1k}}, \boldsymbol{x_{2k}}, \dots, \boldsymbol{x_{nk}})^{\boldsymbol{T}}$, then the entry $\boldsymbol{x_{ik}}$ describes the **probability of being in state i** at time **k**.
- The **transition probabilities** are **fixed** and show how the system progresses in the next time interval. This is equal to $\boldsymbol{x_{k+1}} = \boldsymbol{P x_k}$.

Example:

$$P = \begin{bmatrix} 0.6 & 0.3 & 0 \\ 0.2 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.7 \end{bmatrix}, x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 0.6 & 0.3 & 0 \\ 0.2 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.2 \\ 0.2 \end{bmatrix}$$



- **Definition:** Let P be an $n \times n$ stochastic matrix. A **steady-state** or **equilibrium vector q in** $\mathbb{R}^n$ is a vector, such that

$$\boxed{\boldsymbol{Pq} = \mathbf{1} \cdot \boldsymbol{q} = \boldsymbol{q}.}$$

Note: q is an eigenvector of P for the eigenvalue 1. Every stochastic matrix has a steady state vector.

Proof of Note:

Hint: Can you find an eigenvector v of the transpose P^T of P? What do you know about the columns of P?

MARKOV CHAINS

- **Definition:** Let P be an n × n stochastic matrix. Then P is **regular** if some matrix power P^k contains **no zero entries**.

Theorem 1: (Markov chains) If P be an n × n **regular** stochastic matrix, then P has a **unique** steady-state vector $\boldsymbol{q}$ that is a probability vector. Furthermore, if $\boldsymbol{x_0}$ is **any initial state** and

$$\boldsymbol{x_k} = \boldsymbol{P^k x_0} \quad \text{or equivalently} \quad \boldsymbol{x_k} = \boldsymbol{P x_{k-1}},$$

then the Markov chain

$$(\boldsymbol{x_k})_{\boldsymbol{k}\ \mathbf{in}\ \mathbb{N}} \quad \text{converges to } \boldsymbol{q} \text{ as } k \to \infty.$$

Exercise: Use a computer to find the steady state vector of your mood network.

SEARCH ENGINES

How can we search the web for information?

This part of the presentation is based on the article *"How Google finds your needle in the haystack"* by David Austin.

Goal: Describe Google's PageRank Algorithm.

The web contains more than 30 trillion web pages.

How can we search these pages for information within seconds?

SEARCH ENGINES

What does a search engine do?

1.) **Index web pages:** Search the web and locate all web pages with public access and index the data on these pages.

2.) **Rank the importance of pages:** In order to display the most relevant pages first, it needs to decide which page is most important.

3.)**Match search criteria:** When a user enters one or several keywords, the search engine matches it to the indexed pages with the same keywords. Among these it picks the most important ones and displays them.

SEARCH ENGINES

What does a search engine do?

Google has indexed more than 30 trillion pages. Most of these contain about 10.000 words. This means that there is a huge number of pages that contain the words of a search phrase.

The Big Problem:

Rank the pages such that the important ones are displayed first.

Idea: Model webpages with links as a directed graph. Calculate the importance of a page according to the number of pages linking to it.

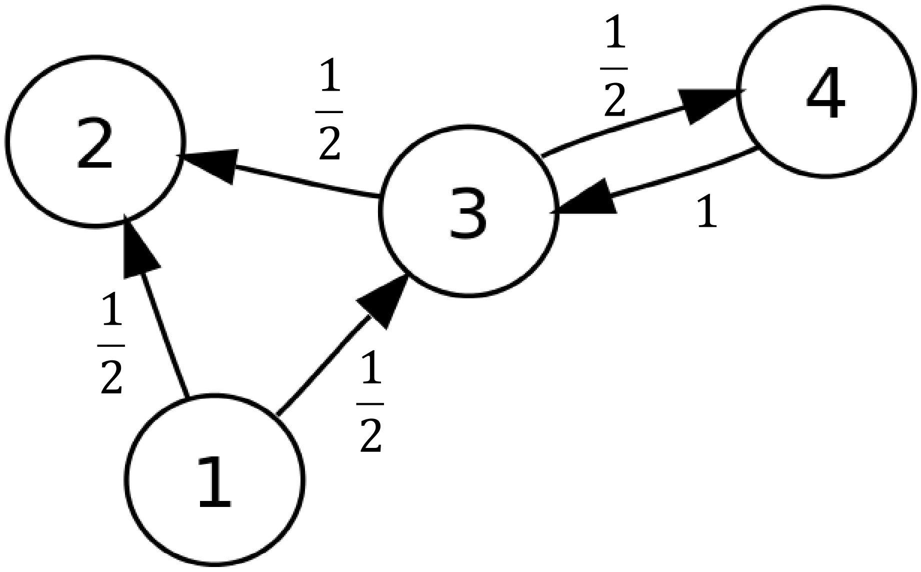
SEARCH ENGINES

Model: We model the web as a directed graph:

- Every vertex $(w_i)_i$ is a webpage.
- Every link from one page to another is an arrow.
- Each webpage distributes its whole value onto the pages its links to with equal weight. Self-links are not counted.

Example:

$$
\begin{array}{c|cccc|c}
\underline{\textbf{From:}} & \textbf{1} & \textbf{2} & \textbf{3} & \textbf{4} & \underline{\textbf{To:}} \\
\hline
 & 0 & 0 & 0 & 0 & \textbf{1} \\
H= & 0.5 & 0 & 0.5 & 0 & \textbf{2} \\
 & 0.5 & 0 & 0 & 1 & \textbf{3} \\
 & 0 & 0 & 0.5 & 0 & \textbf{4}
\end{array}
$$



HYPERLINK MATRIX

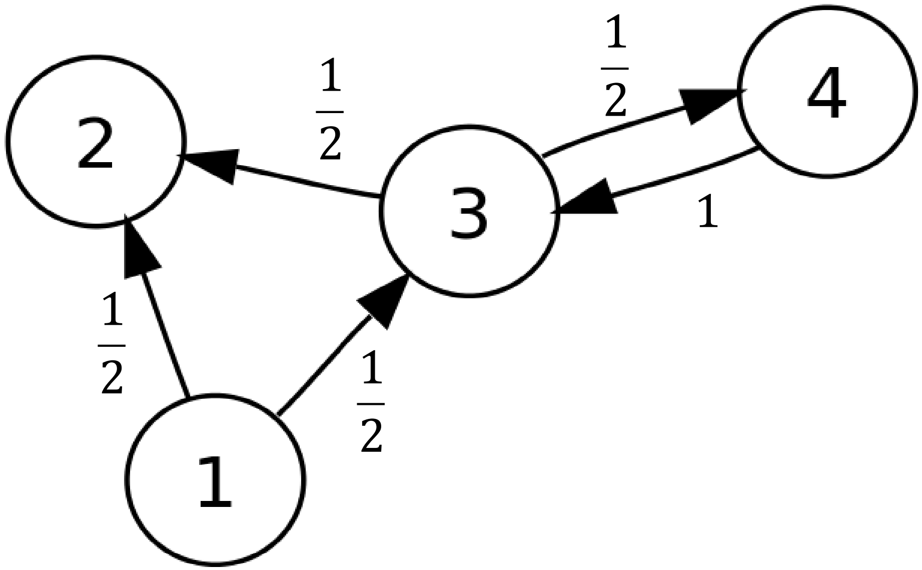
Definition: Let $(w_i)_i$ be the set of webpages on the internet. Let $\boldsymbol{n_i}$ be the **number of pages** the page w_i links to. The **hyperlink matrix** $\boldsymbol{H} = (\boldsymbol{h_{ij}})_{\boldsymbol{i,j}}$ is defined by

$$h_{ij} = \begin{cases} 1/n_j & \text{if} \quad w_j \text{ links to } w_i \\ 0 & \text{otherwise}. \end{cases}$$

Example:

$$\begin{array}{ll} & \begin{array}{cccc} \textbf{1} & \textbf{2} & \textbf{3} & \textbf{4} \end{array} \\ H= & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 1 \\ 0 & 0 & 0.5 & 0 \end{bmatrix} \begin{array}{c} \textbf{1} \\ \textbf{2} \\ \textbf{3} \\ \textbf{4} \end{array} \end{array}$$

From: (columns) **To:** (rows)



HYPERLINK MATRIX

If H was a regular stochastic matrix we could use Theorem 1:

Theorem 1: (Markov chains) If P be an n × n regular stochastic matrix**,** then P has a **unique** steady-state vector $\boldsymbol{q}$ that is a probability vector. If $\boldsymbol{x_0}$ is **any initial state** and $\boldsymbol{x_k} = \boldsymbol{P^k x_0}$ then the Markov chain $(\boldsymbol{x_k})_{\boldsymbol{k} \textbf{ in } \mathbb{N}}$ converges to $\boldsymbol{q}$ as $k \to \infty$.

This means that we could assign the **importance** $\boldsymbol{q_i}$ of a webpage $\boldsymbol{w_i}$ via the steady state vector $\boldsymbol{q} = (\boldsymbol{q_1}, \boldsymbol{q_2}, \dots, \boldsymbol{q_k}, \dots)$, such that

$$\boldsymbol{Hq} = \boldsymbol{q}.$$

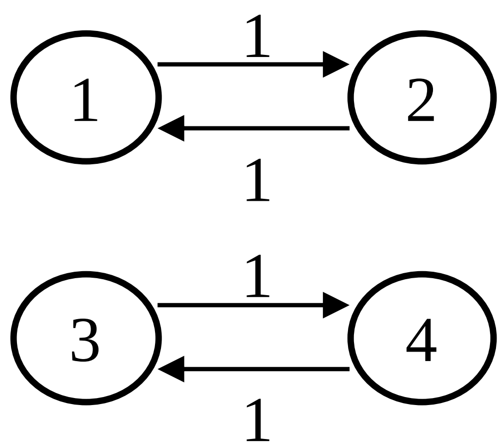
This means that the importance would be the steady-state result of a repeated process of assigning importance.

How can we turn H into a regular stochastic matrix?

Problem 1: Web pages without links (see previous example). In this case there is a **zero column** in the matrix H.

Problem 2: Groups of pages that do not link to other groups.

$$H=\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$



In this case the matrix can **not** be **regular**, as no power of H can have nonzero entries. There would be **no unique steady state vector**.

GOOGLE MATRIX

How can we turn H into a regular stochastic matrix?

Solution: Let **S** be the matrix where zero columns in H are replaced by a column **(1/n,1/n,…,1/n),** where **n** is the number of webpages.

Definition: We define the **Google matrix G** to be

$$\mathrm{G} = 0.85 \cdot \boldsymbol{S} + 0.15 \cdot \frac{1}{n}\mathbf{1}, \quad \text{where}$$

1 is the matrix with all entries equal to 1.

Note: With this definition G is a regular stochastic matrix. We can apply **Theorem 1** and define an importance for each webpage.

How can we define the importance of a webpage ?

The Google matrix G is a regular stochastic matrix. We define

Definition: The **importance** $\boldsymbol{q_i}$ of a webpage $\boldsymbol{w_i}$ is the entry in the steady state vector $\boldsymbol{q} = (\boldsymbol{q_1}, \boldsymbol{q_2}, \dots, \boldsymbol{q_k}, \dots)$, such that

$$\boldsymbol{Gq = q.}$$

We call **q** the **importance vector** of the Google matrix.

Note 1: The steady state vector q can be scaled to obtain a more practical range.

Note 2: In practice the calculation of q is done once per month using the iteration from **Theorem 1**: $\boldsymbol{x_k = Gx_{k-1}}$.