Math 22, Exam II Solutions

May 13, 2010

NAME:

This is a closed book exam and you may not use a calculator. Use the space provided to answer the questions and if you need more space, please use the back of the exam making sure to write a note in the space provided that you have more work elsewhere that you would like me to grade. You must SHOW ALL WORK and be neat. If you have any questions, do not hesitate to ask.

Good luck!

Remember the honor code – do all of your own work.

Warning: Due to having different syllabi, the following topics are not exam questions

- the row space of a matrix
- the change of basis matrix
- the vector space of polynomials
- any proof-based question

Be careful, sometimes parts of a question would still be exam material even if others aren't.

Non-exam questions are denoted by NEX

$$\sim \begin{pmatrix} 1 & 3 & 5 & -6 & 0 & 0 \\ 0 & 1 & 2 & 5 & 0 & 0 \\ 0 & 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 & 19 & 0 & 0 \\ 0 & 1 & 0 & 15 & 0 & 0 \\ 0 & 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -26 & 0 & 0 \\ 0 & 1 & 0 & 15 & 0 & 0 \\ 0 & 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

1. The matrix A has been converted to echelon form as follows:

$$A = \begin{pmatrix} -20 & -59 & -97 & 120 & -219 & -225 \\ 1 & 4 & 8 & -6 & 12 & 48 \\ 1 & 4 & 8 & -6 & 54 & 27 \\ 1 & 4 & 29 & -111 & 96 & 90 \\ 1 & 25 & 29 & 204 & -261 & 132 \\ 22 & 46 & 71 & -237 & 390 & 90 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 3 & 5 & -6 & 11 & 12 \\ 0 & \boxed{1} & 2 & 5 & -7 & 9 \\ 0 & 0 & \boxed{1} & -5 & 6 & 4 \\ 0 & 0 & 0 & 0 & \boxed{2} & 1 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

a. Write down a basis for the row space of A.

NEX

$$\text{Row}(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 5 \\ -6 \\ 11 \\ 12 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 5 \\ -7 \\ 9 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -5 \\ 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

nonzero rows of echelon

b. Write down a basis for the column space of A.

All columns of A except for column 4

c. What is the dimension of Col(A)? $\dim(\text{Col}(A)) = 5$

d. Write down a basis for the null space of A.

By the reduced echelon form a basis is $\begin{pmatrix} 26 \\ -15 \\ 5 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

e. What is the dimension of the subspace of all solutions $\mathbf{x}$ of $A^T\mathbf{x} = 0$?

NEX $\dim(\text{Nul}(A^T)) + \dim(\text{Col}(A)) = 6 \Rightarrow \dim(\text{Nul}(A^T)) = 1$

2. Let

$$A = \begin{pmatrix} 7 & 4 \\ -3 & -1 \end{pmatrix}$$

and let

$$\mathbf{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

a. Show that $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors for A with eigenvalues 5 and 1, respectively.

$$Av = \begin{pmatrix} -14+4 \\ 6-1 \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \end{pmatrix} = 5v \qquad Aw = \begin{pmatrix} -14+12 \\ 6-3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} = 1 \cdot w$$

b. Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

$$P = \begin{pmatrix} -2 & -2 \\ 1 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$$

3. Let A be a 3×3 matrix whose eigenvectors are

$$\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

of eigenvalues 1, -1 and 2 respectively. Find A.

$$P = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}, \quad \left(\begin{array}{ccc|ccc} 0 & 1 & -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 & 2 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 8 & -3 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3/8 & 2/8 & 1/8 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/8 & 6/8 & -1/8 \\ 0 & 1 & 0 & 2/8 & 4/8 & 2/8 \\ 0 & 0 & 1 & -3/8 & 2/8 & 1/8 \end{array}\right)$$

$$\underbrace{\begin{pmatrix} 3/8 & 6/8 & -1/8 \\ 2/8 & 4/8 & 2/8 \\ -3/8 & 2/8 & 1/8 \end{pmatrix}}_{P^{-1}}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Then Compute $A = PDP^{-1} =$

$$= \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \frac{1}{8} \begin{pmatrix} 3 & 6 & -1 \\ 2 & 4 & 2 \\ -3 & 2 & 1 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 0 & -1 & -4 \\ 1 & 0 & 2 \\ -2 & -3 & 0 \end{pmatrix} \begin{pmatrix} 3 & 6 & -1 \\ 2 & 4 & 2 \\ -3 & 2 & 1 \end{pmatrix}$$

$$= \frac{1}{8} \begin{pmatrix} 10 & -12 & -6 \\ -3 & 10 & 1 \\ -12 & -24 & -4 \end{pmatrix}$$

Note: Technically this could appear on the midterm. Practically unless A is 2×2 this wouldn't happen, the computations are too heavy.

4. Let

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

Because A is triangular we can read the eigenvalues from the diagonal

a. What are the eigenvalues of A?

$\lambda_1 = 1, \quad \lambda_2 = 2$

b. What are the algebraic multiplicities of each eigenvalue?

$\lambda_1 = 1$ has multiplicity 2, $\lambda_2 = 2$ has multiplicity 1

c. What are the geometric multiplicities of each eigenvalue?

Geometric mult = # of vectors in the basis for the eigenspace

$$A - I = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ geometric multiplicity of $\lambda_1 = 1$ is 1

$$A - 2I \quad \begin{pmatrix} -1 & -1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} +1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$\Downarrow$

geometric multiplicity of $\lambda_2 = 2$ is 1

d. Is A diagonalizable?

no, because the basis for the 1-eigenspace only has one vector and $\lambda_1 = 1$ has multiplicity 2

5. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}.$$

Observe that $\mathcal{B}$ is a basis for $\mathbf{R}^2$. For $\mathbf{x} = \begin{pmatrix} -7 \\ 8 \end{pmatrix}$ compute $[\mathbf{x}]_{\mathcal{B}}$.

To find $[x]_B$ write the augmented matrix $(b_1\ b_2\ x)$ and row-reduce

$$\begin{pmatrix} 1 & 2 & -7 \\ 1 & -1 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -7 \\ 0 & -3 & 15 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -5 \end{pmatrix} \Rightarrow [x]_B = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

$$\overset{t^2+4t+4}{\overset{\shortparallel}{}}$$

6. The polynomials $\mathcal{B} = \{1, t-2, (t+2)^2\}$ form a basis for $\mathbb{P}_2$. For $\mathbf{x} = 1 + t + t^2$ find $[\mathbf{x}]_{\mathcal{B}}$.

NEX Here the approach of 5) doesn't work.

Let $\mathcal{E} = \{1, t, t^2\}$. Then $[x]_{\mathcal{E}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and

$$[x]_{\mathcal{B}} = \underset{\mathcal{B} \leftarrow \mathcal{E}}{P} [x]_{\mathcal{E}}$$

$$\underset{\mathcal{B} \leftarrow \mathcal{E}}{P} = \left(\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}\right)^{-1} = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \quad \left(\text{b/c } \underset{\mathcal{E} \leftarrow \mathcal{B}}{P} = \left([b_1]_{\mathcal{E}} \; [b_2]_{\mathcal{E}} \; [b_3]_{\mathcal{E}} \right) \right.$$

w/ $b_1 = 1$, $b_2 = t-2$, $b_3 = (t+2)^2$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -4 \\ 0 & 1 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -12 \\ 0 & 1 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\underbrace{\begin{matrix} 1 & 2 & -12 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{matrix}}_{= \left(\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}\right)^{-1}}$$

$$\Rightarrow [x]_{\mathcal{B}} = \underset{\mathcal{B} \leftarrow \mathcal{E}}{P} [x]_{\mathcal{E}} = \begin{pmatrix} 1 & 2 & -12 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -9 \\ -3 \\ 1 \end{pmatrix}$$

Check: $(-9)(1) + (-3)(t-2) + 1\left((t+2)^2\right) =$

$$= -9 - 3t + 6 + t^2 + 4t + 4 = t^2 + t + 1$$

7. Compute the characteristic polynomials of the following matrices.

a.

$$A = \begin{pmatrix} 2 & -2 \\ 1 & 5 \end{pmatrix}.$$

$$\det\begin{pmatrix} 2-\lambda & -2 \\ 1 & 5-\lambda \end{pmatrix} = (2-\lambda)(5-\lambda) + 2 = \lambda^2 - 7\lambda + 12$$

b.

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

$$\det\begin{pmatrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix} = (2-\lambda)\det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = (2-\lambda)(\lambda^2-1)$$

c. For the matrix A, what are the eigenvalues?

$$\lambda^2 - 7\lambda + 12 = (\lambda-3)(\lambda-4) \Rightarrow \lambda_1 = 3,\ \lambda_2 = 4$$

or $\lambda_{1,2} = \dfrac{7 \pm \sqrt{49 - 4(12)}}{2} = \dfrac{7 \pm 1}{2} = \begin{cases} 8/2 = 4 \\ 6/2 = 3 \end{cases}$

d. For the matrix B, what are the eigenvalues?

$$(2-\lambda)(\lambda^2-1) = (2-\lambda)(\lambda+1)(\lambda-1) \Rightarrow \lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = -1$$

8. True or false:

a. The only eigenvalue of the 0 matrix is 0.

True : $0x = \lambda x$ for nonzero $x \Rightarrow -\lambda I x = -\lambda x = 0 \Rightarrow \lambda = 0$

b. 7 is an eigenvalue of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 7 \\ 0 & 0 & 1 \end{pmatrix}.$$

False, eigenvalues are 1, 1, 3

c. The sum of two diagonal matrices is a diagonal matrix.

True

d. The set of polynomials of the form $2t - at^2 + bt^3$, where a and b are arbitrary real numbers is a subspace of $\mathbb{P}_3$.

NEX

False 0 is not in it

e. If A is a 7×8 matrix having rank 4, then its null space is 4 dimensional.

True : Rank + dim(Nul) = # Columns

f. A matrix A having distinct eigenvalues is invertible.

False A is invertible if and only if zero is not an eigenvalue

9. Show that if λ is an eigenvalue for A, then 2λ is an eigenvalue for $2A$.

Possible bonus question

If $Ax = \lambda x$ for some x,

then $2Ax = 2(Ax) = 2(\lambda x) = (2\lambda)x$ so

2λ is an eigenvalue for $2A$ with

the same eigenvector

10. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}, \qquad \text{and} \qquad \mathcal{C} = \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

NEX

Find the change of basis that converts an element in $\mathcal{B}$ coordinates to an element in $\mathcal{C}$ coordinates (usually denoted by $P_{\mathcal{C}\leftarrow\mathcal{B}}$).

By Section 4.7, $\left(c_1 \; c_2 \mid b_1 \; b_2 \right) \sim \left(I \mid P_{\mathcal{C}\leftarrow \mathcal{B}} \right)$

$$\left(\begin{array}{cc|cc} -1 & 1 & 5 & 2 \\ 0 & 1 & 3 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} -1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 1 \end{array}\right)$$

$$\Rightarrow P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}$$