1. Determine whether each statement below is true or false and indicate your answer by circling the appropriate choice (1pt each):

(a) (True / False) Let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix such that $AB = O$ (where O represents the $m \times p$ zero matrix). Then, the columns of B are in Nul$A$.

(b) (True / False) Let $\mathbb{P}_n$ denote the vector space of polynomials $p(x)$ of degree at most n. The set of all polynomials in $\mathbb{P}_n$ with $p(0) = 1$ is not a subspace of $\mathbb{P}_n$.

(c) (True / False) Suppose A is a 5×5 matrix with exactly 3 distinct eigenvalues. Suppose further that two eigenspaces of A are 2-dimensional. It is possible that A is not diagonalizable.

(d) (True / False) Suppose $B_3 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 1 \\ 0 & 0 & c \end{bmatrix}$ is an echelon form of A obtained through the following series of elementary row operations: B_1 is obtained by interchanging two rows of A; B_2 is obtained from B_1 by performing a row replacement; and lastly, a scaling of each row is performed so that $B_3 = \frac{1}{5}B_2$. Then $det(A) = -5^3abc$.

(e) (True / False) If $\mathbf{v}_1$ is an eigenvector of A corresponding to λ_1 and $\mathbf{v}_2$ is an eigenvector of A corresponding to λ_2, where λ_1 and λ_2 are distinct eigenvalues of A, then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

(f) (True / False) Any linearly independent set in a subspace H is a basis for H. The set must also span H to be a basis.

(g) (True / False) If A is a 4×3 matrix whose null space has dimension 2, then A can have rank 2. Rank Thm $\Rightarrow$ rank $A = 3 - \dim \text{Nul} A = 3 - 2 = 1$.

(a) $AB = A[\vec{b}_1 \cdots \vec{b}_p] = [A\vec{b}_1 \cdots A\vec{b}_p] = [\vec{0} \cdots \vec{0}] = O$
$\Rightarrow A\vec{b}_i = \vec{0}$, $i = 1, \ldots, p$ $\Rightarrow$ each $\vec{b}_i$ in Nul$A$.

(b) $\{p(x) \in \mathbb{P}_n \mid p(0) = 1\} = \{p(x) = 1 + a_1x + a_2x^2 + \cdots + a_nx^n \mid a_i \in \mathbb{R}\} \subseteq \mathbb{P}_n$ does not contain the zero vector of the vector space $\mathbb{P}_n$. (Alternatively, the set is not closed under addition or scalar mult.)

(c) From the two 2-dimensional eigenspaces we get 4 linearly independent eigenvectors; from the third eigenspace, we can find at least one more eigenvector that is linearly independent from the other 4. That is, one can select $2+2+1 = 5$ eigenvectors of A that form a linearly independent set and hence form a basis for $\mathbb{R}^5$ by the Basis Thm. Hence, A must be diagonalizable.

(d) $\det A = \det B_1 = -\det B_2$ and $\det B_3 = \left(\frac{1}{5}\right)^3 \det B_2 \Rightarrow \det B_2 = 5^3 \det B_3 = 5^3abc$
$\Rightarrow \det A = -5^3abc$.

Note: Other solns are possible

2. Let $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 3 & 7 & 10 & 1 \end{bmatrix}$. Determine a basis for the following subspaces:

(a) Col$A$ (3pts)

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \\ 3 & 7 & 10 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 2 & 3 & 0 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

The pivot columns of A form a basis for ColA, hence $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} \right\}$ is a basis for ColA.

(b) Row$A$ (2pts)

The nonzero rows of an echelon form of A form a basis for RowA.

Hence, $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for RowA.

(c) Nul$A$ (2pts) $\vec{x} \in$ NulA iff $\vec{x}$ is a soln to $A\vec{x} = \vec{0}$.

$$B \sim \begin{bmatrix} \boxed{1} & 0 & 1 & -2 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow x_3 + x_4 \text{ are free variables.}$$

So, solns to $A\vec{x} = \vec{0}$ have the form

$$\vec{x} = \begin{bmatrix} -x_3 + 2x_4 \\ -x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, a basis for NulA is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

3. Suppose $H = Span\{\mathbf{e}_1\}$, $K = Span\{\mathbf{e}_2\}$, where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the standard basis for $\mathbb{R}^2$.

(a) Explain why H and K are subspaces of $\mathbb{R}^2$. (2pts)

Since $\vec{e}_1, \vec{e}_2 \in \mathbb{R}^2$ and $\mathbb{R}^2$ is a vector space, Span$\{\vec{e}_1\}$ and Span$\{\vec{e}_2\}$ are subspaces of $\mathbb{R}^2$. (see Thm 1 §4.1).

(b) Is the intersection of H and K ($H \cap K$) a subspace of $\mathbb{R}^2$? Explain. (2pts)

$H \cap K = \{\vec{0}\} \subseteq \mathbb{R}^2$. The set containing only the zero vector of a vector space is a subspace of the vector space.
(see example 6 §4.1).

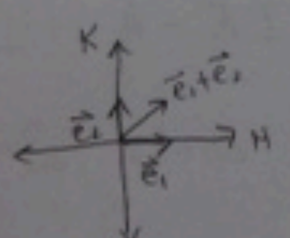
(c) Is the union of H and K ($H \cup K$) a subspace of $\mathbb{R}^2$? Explain. (2pts)

No, because $H \cup K$ is not closed under vector addition.

For example, $\vec{e}_1 \in H$, + so $\vec{e}_1 \in H \cup K$. Likewise, $\vec{e}_2 \in K$ and so $\vec{e}_2 \in H \cup K$.

But, $\vec{e}_1 + \vec{e}_2 = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \notin H \cup K$

(because $\begin{bmatrix}1\\1\end{bmatrix}$ is in neither H nor K):



4. Determine whether the set $\mathbb{B} = \{p_1, p_2, p_3\}$ is a basis for $\mathbb{P}_2$ (the set of all polynomials of degree at most 2), where $p_1(x) = 3x^2 + x + 1$, $p_2(x) = 2x + 1$, and $p_3(x) = 2$. Fully justify your answer. (9pts)

Let $\mathcal{C} = \{1, x, x^2\}$. $\mathcal{C}$ is the standard basis for $\mathbb{P}_2$. The elements of $\mathbb{P}_2$ have the form $p(x) = a_0 + a_1 x + a_2 x^2$. Thus, $[p(x)]_C = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$.

Let $A = \left[[p_1]_C \; [p_2]_C \; [p_3]_C \right]$.

Then, $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$.

By the IMT, the <u>columns of A form a basis for $\mathbb{R}^3$</u>.

Because $[\;]_C$ is an <u>isomorphism</u> from $\mathbb{P}_2$ onto $\mathbb{R}_3$,

<u>B forms a basis for $\mathbb{P}_2$</u>.

5. (a) Suppose that an $n \times n$ matrix A has a zero eigenvalue. Explain why A must be a singular matrix. (1pt)

$\lambda = 0$ an eigenvalue of $A \Rightarrow \det(A - \lambda I) = 0$

$\Rightarrow \det A = 0$

$\Rightarrow A$ is singular.

(b) For the remaining parts of this problem, let $A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$. Determine the eigenvalue(s) of A. (2pts)

$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 2 & -2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda) + 2 = (\lambda-1)(\lambda+2)+2$

$= \lambda^2 + \lambda - 2 + 2 = \lambda^2 + \lambda = \lambda(\lambda+1)$.

$\Rightarrow$ the eigenvalues of A are 0 and -1.

(c) Determine a basis for each eigenspace of A. (4pts)

A basis for the eigenspace corresponding to the eigenvalue λ of A is a basis for $\text{Nul}(A - \lambda I)$.

$\underline{\lambda = 0}$: $A - \lambda I = A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ Elements of $\text{Nul} A$ have the form $\vec{x} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, x_2 free.

$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of A corresponding to $\lambda = 0$.

$\underline{\lambda = -1}$: $A - \lambda I = A + I = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$.

So, elmts of $\text{Nul}(A+I)$ have the form $\vec{x} = \begin{bmatrix} 1/2 x_2 \\ x_2 \end{bmatrix}$

$= x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$, x_2 free.

Thus $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ forms a basis for the eigenspace of A corresponding to $\lambda = -1$.

(d) Explain why A is diagonalizable. (1pt)

A has 2 linearly independent eigenvectors.
Thus, A has enough lin. indep. eigenvectors to span $\mathbb{R}^2$.
$\therefore$ A is diagonalizable.

(e) Use the fact that A is diagonalizable to calculate A^{1000}. (7pts)

$A = PDP^{-1}$, where $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.

$A = PDP^{-1} \Rightarrow A^{1000} = PD^{1000}P^{-1}$.

$D^{1000} = \begin{bmatrix} 0 & 0 \\ 0 & (-1)^{1000} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

$P^{-1} = \frac{1}{(1)(2)-(1)(1)} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

So, $A^{1000} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$

6. Find a basis for $H = \left\{ \begin{bmatrix} a+2b-4c \\ -5b+15c \\ a+b-c \\ a+b+3c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$. (6pts)

$$\begin{bmatrix} a+2b-4c \\ -5b+15c \\ a+b-c \\ a+b+3c \end{bmatrix} = a \overbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}}^{=\vec{v}_1} + b \overbrace{\begin{bmatrix} 2 \\ -5 \\ 1 \\ 1 \end{bmatrix}}^{=\vec{v}_2} + c \overbrace{\begin{bmatrix} -4 \\ 15 \\ -1 \\ 3 \end{bmatrix}}^{=\vec{v}_3} \in H \quad \forall a, b, c \in \mathbb{R}.$$

Because all elements of H can be written as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$, we know that $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ spans H. To find a basis for H, we must find a subset of S (possibly S itself) that spans H <u>and</u> is linearly independent.

Let $A = [\vec{v}_1, \vec{v}_2, \vec{v}_3] = \begin{bmatrix} 1 & 2 & -4 \\ 0 & -5 & 15 \\ 1 & 1 & -1 \\ 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$ (*)

Note that $H = \text{Col} A$, so a basis for Col A is a basis for H.

Every column of A is a pivot column, hence $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for H.

(Alternatively, (*) shows that the cols of A form a lin. indep set. Thus, S both lin. indep. + a spanning set for H $\Rightarrow$ S is a basis for H).