SOLUTIONS

Barnett 8/28/17

Your name:

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Math 22 Summer 2017, Final, Sunday Aug 27 / Monday Aug 28

Please show your work. No credit is given for solutions without work or justification.

1. [9 points] Consider the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ in REF}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \sim \underset{\text{REF}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}$$

2pts. (a) Which of the matrices A, B are invertible?

1pt for each

neither. (both have a free variable)

3pts (b) Which of the matrices A, B have an eigenspace of dimension 2? A only:

A has $\lambda = 0, 1$ (twice), reading eigvals from diagonal (since it's upper-triangular)

$A - 1I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ 2 free vars $\Rightarrow \dim \text{Nul}(A - 1I) = 2$.

+2 pts for correct work but incorrect interpretation i.e. "B has dim 1+1 = 2"...

B: $\underbrace{(1-\lambda)\left[(1-\lambda)(-\lambda) - 1 \cdot 0\right]}_{\det(B - \lambda I)} = 0$ so $\lambda = 0, 1$ (twice), same as A.

But $B - 1I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

so $\dim \text{Nul}(B - 1I) = 1$.

1pt (c) Which of the matrices A, B are diagonalizable?

A only.

3pts (d) Diagonalize every diagonalizable matrix from the previous part (i.e. find a diagonal D and invertible P so that the diagonalizable matrix equals PDP^{-1}. Do not compute P^{-1}). We diagonalize A:

$\lambda_{1,2} = 1$: $A - 1I \sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_3 = 0$: $A - 0I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ so $\begin{matrix} x_1 = 0 \\ x_2 = -x_3 \\ x_3 = x_3 \end{matrix}$ so $\vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

free

stack in order:

$$D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

or any corresponding column reordering.

1pt for D. 2pts for P.

2. [11 points] Consider the following web with three pages and links given by the diagram:

$$a \longrightarrow b \longleftarrow c$$

2pts (a) Let A be the stochastic matrix for this web given by the PageRank algorithm (with the usual $\alpha = 1$). Find A, using the ordering a, b, c.

+1 for $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$A = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1/3 & 0 \\ 1 & 1/3 & 1 \\ 0 & 1/3 & 0 \end{bmatrix} \end{array}$$

since b has no outgoing links, its column is replaced by all $1/3$, ie jumping to a random webpage. This is part of PageRank.

3pts (b) Find the vector of importances for this web. Write this vector as a probability vector.

Solve $A\vec{x} = \vec{x}$, ie $\vec{x}$ is eigenvector w/ $\lambda = 1$:

+1 for
+1 for
+1 for

$$A - I = \begin{bmatrix} -1 & 1/3 & 0 \\ 1 & -2/3 & 1 \\ 0 & 1/3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & -1/3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

x_3 free

+2 total for rescaling by $\|\cdot\|_2$

so $x_1 = x_3$, $x_2 = 3x_3$, $x_3 = x_3$ $\Rightarrow \vec{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ $\xrightarrow[\text{sum is } 1]{\text{scale so}}$ $\vec{q} = \begin{bmatrix} 1/5 \\ 3/5 \\ 1/5 \end{bmatrix}$

4pts (c) Find a diagonal matrix D and an invertible matrix P so that $A = PDP^{-1}$. [Do not compute P^{-1}.]

+1 for D
+3 for P
• 1 for each eigen vector

We computed eigenpair $\lambda_1 = 1$, $\vec{v_1} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, above.

Since A has two rows identical, $\lambda_2 = 0$ is also an eigenvalue.

$$A \sim \begin{bmatrix} 1 & 1/3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ so $x_1 = -x_3$, $x_2 = 0$, $x_3 = x_3$, so $\vec{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Char eqn: $-\lambda\left[(1/3 - \lambda)(-\lambda) - 1/3\right] - \frac{1}{3}\left[-\lambda - 0\right]$

$$= -\lambda\left[\lambda^2 - \frac{1}{3}\lambda - \frac{2}{3}\right] = -\lambda(\lambda - 1)(\lambda + 2/3) = 0$$

So $\lambda_3 = -2/3$: $A + \frac{2}{3}I = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 1 & 1 & 1 \\ 0 & 1/3 & 2/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

So $D = \begin{bmatrix} 1 & & \\ & 0 & \\ & & -2/3 \end{bmatrix}$, $P = \begin{bmatrix} 1 & -1 & 1 \\ 3 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}$, or any corresponding column reordering.

2pts. (d) Must the Markov chain for A converge to your answer from (b), regardless of its initial probability vector ? Explain.

Yes. Two ways to prove this:

+1 for Yes
+1 for explanation

i) $\lambda_1 = 1$ but $|\lambda_2|, |\lambda_3| < 1$, so $\vec{x}^{(k)} = PD^kP^{-1}\vec{x}^{(0)}$ tends to $\vec{q}$, because $D^k \to \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}$ as $k \to \infty$.

OR, ii) A is regular since $A^2 = \begin{bmatrix} 1/3 & 1/9 & 1/3 \\ 1/3 & 7/9 & 1/3 \\ 1/3 & 1/9 & 1/3 \end{bmatrix}$ has strictly positive entries.

3. [7 points] Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

$\vec{x}_1 \quad \vec{x}_2 \quad \vec{x}_3$

4pts. (a) Find an *orthogonal* basis for Col A.

Use Gram-Schmidt on columns of A:

+2 for 2/3 correct vectors and minor numerical mistake

$\vec{v}_1 = \vec{x}_1$

$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \vec{x}_2.$

$=0$ since $\vec{x}_2$ already orthog. to $\vec{x}_1$.

Prepare $\begin{bmatrix} 2 \\ 3 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = 6$

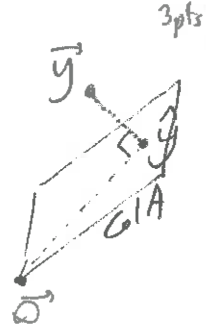
$\begin{bmatrix} 2 \\ 3 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = -3$

$\vec{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -2 \end{bmatrix} - \frac{6}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{(-3)}{3}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$

check $\vec{v}_3 \cdot \vec{x}_1 = \vec{v}_3 \cdot \vec{x}_2 = 0$ ✓

basis is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$

3pts. (b) Find the coordinates of the point in Col A that is closest to the point $\begin{bmatrix} 0 \\ -2 \\ 5 \\ 4 \end{bmatrix} = \vec{y}$



Since have orthog basis $\vec{v}_1 \cdots \vec{v}_3$ above, easy.

By best approximation theorem, closest pt. is $\hat{y} = \text{proj}_{\text{Col} A}\vec{y}$:

$\hat{y} \overset{\star}{=} \frac{\vec{y} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \frac{\vec{y} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 + \frac{\vec{y} \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3}\vec{v}_3$

+2 for numerical mistake but correct

+1 for $QQ^T\vec{y}$ but Q without orthonormal columns.

$= \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \frac{6}{3}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \frac{-9}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \\ 5 \end{bmatrix}$

+1 for using least-squares but incorrectly (must produce a vector $\mathbb{R}^4$)

+2 for $QQ^T\vec{y}$ with correct Q but numerical →

+1 for using A with non orthogonal basis!

one slip allowed in (b) or (c) but not (a), since that's trivial.

4. [8 points] Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$.

A^TA

4 pts. (a) Solve the inconsistent system $A\mathbf{x} = \mathbf{b}$ in the least-squares sense.

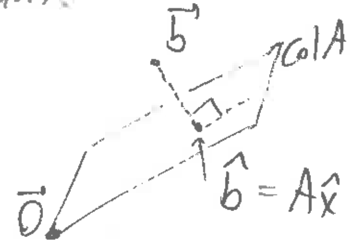
Set up "normal equations": $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$

& $A^T\vec{b} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Solve them: $(A^TA)\hat{x} = A^T\vec{b}$, ie

$$\left[\begin{array}{cc|c} 2 & 2 & 4 \\ 2 & 5 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 3 & -3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \end{array}\right]$$

$\hat{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is the unique least-squares solution.



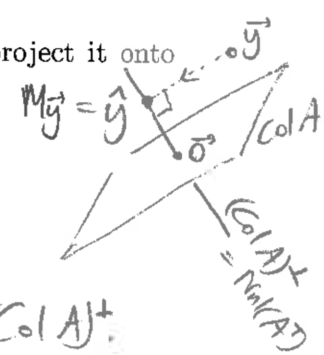
2 pts. (b) What is the smallest possible value of $\|A\mathbf{x} - \mathbf{b}\|$ for any $\mathbf{x} \in \mathbb{R}^2$?

By best approx. theorem, $\vec{x} = \hat{x}$ minimizes $\|A\vec{x} - \vec{b}\|$.

$$\text{So } \|A\hat{x} - \vec{b}\| = \Big\| \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix}}_{\hat{b} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}} - \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \Big\| = \Big\| \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \Big\|$$

$= \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$.

2 pts. (c) Write a matrix whose action on any vector in $\mathbb{R}^3$ is to orthogonally project it onto $(\text{Col } A)^{\perp}$. [You may use a factored form to avoid writing all 9 entries.]



Get a basis for $\text{Nul}(A^T)$: $A^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$ free

so $x_1 = -x_3$, $x_2 = 2x_3$, $x_3 = x_3$, ie $\vec{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ spans $\text{Nul } A^T = (\text{Col } A)^{\perp}$.

normalize to $\vec{u} = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$, ← 1pt.

Then $M_{\perp} = \vec{u}\vec{u}^T = \begin{bmatrix} 1/6 & -2/6 & -1/6 \\ -2/6 & 4/6 & 2/6 \\ -1/6 & 2/6 & 1/6 \end{bmatrix}$ is the projector.

↑ outer product form is ok.

5. [7 points]

2pts (a) Is $W = \left\{ \begin{bmatrix} s \\ t \\ t \end{bmatrix} : s, t \text{ real} \right\}$ a vector space? Prove your claim as $\boxed{\text{succinctly}}$ as possible.

+1 if verify all axioms of subspace

+1 if write "W is span of single vector"

0 for no explanation or insufficient explanation

$W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$, and any span is a subspace, therefore a vector space in its own right, so, yes.

3pts (b) Every element of the above W is in the span of $\left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\}$. Is this set a basis for W ? Explain.

+2 for vague explanation but correct answer and idea.

Tests for being a basis:

i) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent? yes, it is.

ii) is W equal as a set to $\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$? No, $W \subset \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3$, but $W \neq \mathbb{R}^3$.

Note $\vec{v}_1 \in W$, but $\vec{v}_2$ and $\vec{v}_3$ are <u>not in W</u>, so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ cannot be spanning set for W.

2pts (c) Is the set $H := \{x \in \mathbb{R} : x \geq -1\}$ a vector space? Explain why.

No. $H \subset \mathbb{R}$, so check subspace tests: a) H contains 0? yes.

b) x, y in $H \Rightarrow x+y$ in H? Not true for $x = y = -1$, since -2 is not in H. <u>OR</u> c) cx not in H for $x = 1, c = -2$.

6. [8 points] Short ones.

3pts. (a) For what real values of a is the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & a \\ 0 & 1 & a \end{bmatrix}$ invertible?

Fortunately, a never appears in a pivot other than the last one, so there are no complicated cases to consider.

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & a-1 \\ 0 & 1 & a \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1-a \\ 0 & 0 & 2a-1 \end{bmatrix}$$

invertible if $2a-1$ is a pivot, ie $2a-1 \neq 0$

$\Rightarrow a \neq 1/2$ it's invertible

3pts. (b) An engineering problem produces A, a 2017×2019 matrix, known to possess three linearly independent column vectors that it multiplies to give the zero vector. Must there be a right-hand side $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution? Explain.

$n=2019$

$m=2017$ $\boxed{A}$

ie, $A\vec{x_j} = \vec{0}$ for $j=1,2,3$, $\{\vec{x_1}, \vec{x_2}, \vec{x_3}\}$ L.I. set.

$\hookrightarrow$ this means $\dim \operatorname{Nul} A \geq 3$.

By the rank theorem, $\operatorname{rank} A + \dim \operatorname{Nul} A = 2019 = n$

so $\operatorname{rank} A \leq 2019 - 3 = 2016$. There cannot be a pivot in each of the 2017 rows. $\Rightarrow$ not consistent for all $\vec{b}$.

$\Rightarrow$ There must be a $\vec{b}$ for which incons.

2pts. (c) If a 5×5 matrix has characteristic polynomial of the form $-\lambda^3(\lambda^2 + a\lambda + b)$, what are the possible values for its rank?

$\lambda = 0$ (3 times)

could have $\lambda=0$ also a root, 0, 1, or 2 times!

From the char. poly, $\overbrace{\dim \operatorname{Nul} A}^{=5-\operatorname{rank}A \text{ by Thm.}} \geq 1$

since $\lambda = 0$ is an eigenvalue.

Although the $\lambda=0$ is multiplicity 3, b (ka) could $=0$ too, so Rank is 4, 3, 2, 1, or 0.

even if (as stated) graded correct.

Also correct: If $b \neq 0$, you can say rank is 4, 3, 2, only.

7. [9 points]

3 pts. (a) Let A be an $n \times n$ matrix. Prove that the product of its eigenvalues equals its determinant. [Hint: write its characteristic polynomial two ways.]

1pt for this. $\det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$

where $\lambda_1, \cdots \lambda_n$ are the n eigenvalues counting multiplicities.

This is true for any λ.

So, you can simply set $\lambda = 0$, to give

$$\det A = \lambda_1 \cdots \lambda_n = \prod_{j=1}^{n} \lambda_j \qquad \square$$

Note: partial credit (2pts) if assumed (incorrectly) that A was diagonalizable, then used $\det A = \det PDP^{-1} = (\det P) \det D (\det P^{-1})$.

1pt if only did 2x2 case, or diag. case. Note: $\det(A - \lambda I) \neq \det A - \det(\lambda I)$ in general !!

3pts (b) Let A be any $m \times n$ matrix. Prove that, if there exists a matrix B such that $BA = I$, where I is the $n \times n$ identity matrix, then A has linearly independent columns.

Note A is $m \times n$, generally not square, so can't use the I.M.T. (partial credit if assumed this).

Proof: let $\vec{x}$ solve $A\vec{x} = \vec{0}$. If we can show $\vec{x} = \vec{0}$, then by definition of linear independence, we're done.

Take $A\vec{x} = \vec{0}$ and left-multiply both sides by B:

$$\underbrace{BA}_{I}\vec{x} = B\vec{0} = \vec{0}. \qquad \text{So } I\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}.$$

by hypothesis given. $\square$

3pts (c) Again let A be any $m \times n$ matrix. Prove that if A has linearly independent columns, then there exists a matrix B for which $BA = I$. [This is the converse of part (b), so together they prove: having L.I. columns is *equivalent* to possessing a "left inverse".]

unless you only considered $m = n$, square, in which case IMT.

This is tricky. i) One such proof is constructive, ie, it actually supplies a left inverse B. Namely:

Since A has L.I. columns, it has a QR decomposition,

ie $\underset{m\times n}{\boxed{A}} = \underset{m\times n}{\boxed{Q}}\,\underset{n\times n}{\boxed{R}}$ with Q's columns orthonormal, and R invertible (& upper triangular).

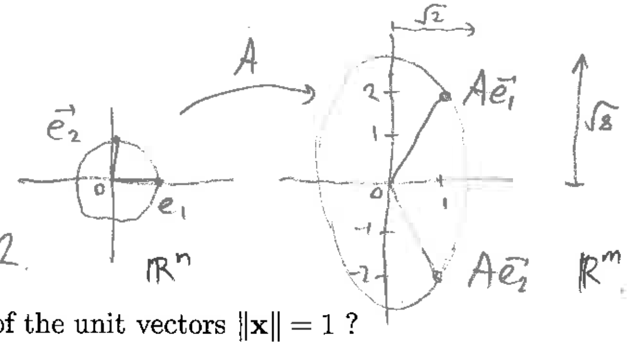
Let $B = R^{-1}Q^T$. Then $BA = R^{-1}\underbrace{Q^TQ}_{I}R = R^{-1}R = I$. $\square$

OR ii) Another pf (adapted from Han Cai): rank A^T = rank $A = n$, so $A^T\vec{x} = \vec{e}_j$ has a solution, $\vec{x}_j$, for each unit vector $j = 1, \cdots n$. Stack $B = \begin{bmatrix} \vec{x}_1^T \\ \vdots \\ \vec{x}_n^T \end{bmatrix}$. $\square$

+1 BONUS: to what property is possessing a "right inverse" equivalent?

to having a pivot in every row, ie that Col $A = \mathbb{R}^m$, or rank $A = m$.

Why? take Transpose of arguments above, & recall rank A^T = rank A.



8. [10 points] Consider the matrix $A = \overset{\xleftrightarrow{n=2}}{\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}}$. $\updownarrow m=2$

4pts. (a) What is the maximum $\|A\mathbf{x}\|$ can have over all of the unit vectors $\|\mathbf{x}\| = 1$?

This is σ_1, the largest singular value of A.

$$A^TA = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$$

$$\begin{vmatrix} 5-\lambda & -3 \\ -3 & 5-\lambda \end{vmatrix} = 25 - 10\lambda + \lambda^2 - 9 \quad \text{ie } \lambda_1 = 8, \ \lambda_2 = 2$$
$$= (\lambda - 8)(\lambda - 2)$$

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{8}.$$

1pt (b) State a single geometric interpretation of the *smallest singular value* of A.

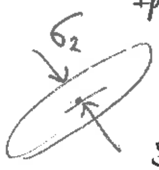


image of $A\vec{x}$ for $\|\vec{x}\| = 1$.

Length of semi-minor axis of ellipse, OR, Min. dist. of any $A\vec{x}$ from origin, for $\|\vec{x}\| = 1$.

5pts (c) Compute the full singular value decomposition of A (i.e. give U, Σ and V. Choose signs so that the top row of V has positive entries.) don't penalize.

$$A^TA - 8I = \begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{so} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$A^TA - 2I = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{so} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$\vec{u}_1 =$ normalized $A\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ easy

$\vec{u}_2 =$ " $A\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

so

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

SVD is:

$$\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{V^T}$$

spanned by $\begin{bmatrix}1\\1\end{bmatrix}$ spanned by $\begin{bmatrix}1\\0\end{bmatrix}$

9. [11 points] In this question only, no working is needed; just circle T or F.

(a) T / F: If A is row-equivalent to B, then $\operatorname{Col} A = \operatorname{Col} B$. viz. $\begin{bmatrix}1&0\\1&0\end{bmatrix} \sim \begin{bmatrix}1&0\\0&0\end{bmatrix}$

[don't confuse w/ $\operatorname{Row} A = \operatorname{Row} B$].

(b) T / F: If the linear system $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution, then $\mathbf{b}$ can be written as a linear combination of the columns of A.

having LSQ soln. doesn't imply consistent.

(c) T / F: If two vector spaces are isomorphic, they must have the same dimension.

is a very useful consequence of isomorphism.

(d) T / F: If a matrix A is diagonalizable, then every eigenvalue of A has algebraic multiplicity equal to 1.

don't confuse w/ converse, which is true.

viz. $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ is diag'ble.

(e) T / F: The set of continuous functions $f(t)$ on $0 \le t \le 1$ obeying $\int_0^1 f(t)dt = 0$ is a vector space.

the condition of zero integral is linear. Check $\int f+g \, dt = 0$ if f, g in space, etc.

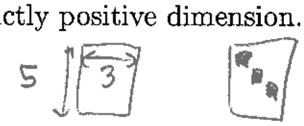
(f) T / F: Let A be an $n \times n$ matrix. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of $\mathbb{R}^n$ consisting of eigenvectors of A. If we apply the Gram-Schmidt algorithm to $\mathcal{B}$, then the result is an orthogonal basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

no, mixing up different eigenspaces V with G-S generally destroys eigenvectors.

(g) T / F: The subset $\{1+2t^3, 2+t-3t^2, -t+2t^2-t^3\}$ is a linearly independent subset of $\mathbb{P}_3$.

Check $\begin{bmatrix}1&2&0\\0&1&-1\\0&-3&2\\2&0&-1\end{bmatrix}$ has pivot in every column, or $\vec{v}_3 \notin \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$

(h) T / F: If A is a 5×3 matrix with rank 3, then it is impossible for $\operatorname{Nul} A$ to have strictly positive dimension.



$\dim \operatorname{Nul} A = n - \operatorname{rank} A = 3 - 3 = 0$.

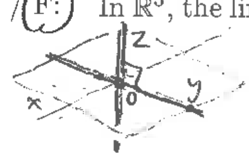
(i) T / F: The zero vector is the only vector in $\mathbb{R}^n$ that satisfies $\|\mathbf{v}\| = \mathbf{v} \cdot \mathbf{v}$.

$\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$ so $\|\vec{v}\| = \|\vec{v}\|^2$ so $\|\vec{v}\| = 0$, so $\vec{v} = \vec{0}$.

(j) T / F: Whenever A is an orthogonal matrix, so is A^3. A is square.

Check $(A^3)^T A^3 = \cancel{A^T}\cancel{A^T}\cancel{A^T}\cancel{A}\cancel{A}\cancel{A} = I$

(k) T / F: In $\mathbb{R}^3$, the line $x = y = 0$ and the line $x = z = 0$ are orthogonal complements.

$\mathbb{R}^3$: 

As sets they are mutually orthogonal, but $\{x=y=0\}^\perp$ = the whole xy plane.