# Math 22: Final Exam

November 16, 2012, 3pm-6pm

Your name (please print): ______________________________________________

**Instructions**: This is a closed book, closed notes exam. **Use of calculators is not permitted**. Unless otherwise stated, you must justify all of your answers to receive credit - please write in complete sentences in a paragraph structure. You may not give or receive any help on this exam and all questions should be directed to Professor Pauls.

You have **3 hours** to work on all **9** problems. Please do all your work in this exam booklet.

**The Honor Principle requires that you neither give nor receive any aid on this exam.**

| Problem | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 35 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 5 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 10 | |
| Total | **100** | |

(1) (10 points) Complete the following definitions - remember, state definitions of the terms, not properties of the terms. To get credit, your answers must make sense as English sentences.

(a) A set of vectors is linearly independent if ...

(b) A map $T : V \to W$ is a linear transformation of vectors spaces if ...

(c) A matrix $A$ is invertible if ...

(d) Let $B$ be an $n \times n$ matrix. Then, a vector $\vec{v}$ is an eigenvector of $A$ if ...

(e) A set of vectors $\mathfrak{B} = \{\vec{v}_1, \ldots, \vec{v}_k\} \subset V$ is a basis for the vector space $V$ if ...

(f) A matrix $C$ is an orthogonal matrix if ...

(g) A a Markov chain is …

(h) Let $D$ be a square matrix. Then, the characteristic polynomial of $D$ is …

(i) The rank of a matrix is …

(j) The least squares solution to the matrix equation $A\vec{x} = \vec{b}$ is ...

(2) (35 points total, 5 points each) For each question, explain your process and write clearly. All answers must be fully justified, especially answers to yes or no questions.

(a) Let $A_1 = \begin{pmatrix} 1 & 2 & -4 & -4 \\ 2 & 4 & 0 & 0 \\ 2 & 3 & 2 & 1 \\ -1 & 1 & 3 & 6 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 5 \\ 2 \\ 5 \\ 5 \end{pmatrix}$. Find all solutions to the matrix equation $A_1\vec{x} = \vec{b}$ or show that no solutions exist.

(b) Let $A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 0 & -1 & 2 \end{pmatrix}$. Show that the columns of $A_2$ are either linearly dependent or linearly independent. What does this say about the dimension of $Col\ A$? Does this imply anything about the dimension of $Nul\ A$? If so, what and why?

(c) Let $A_3 = \begin{pmatrix} 2 & 3 \\ 1 & 5 \\ 4 & 7 \\ 3 & 6 \end{pmatrix}$. Find a basis for $Nul\ A_3$. What is the rank of $A_3$? Is $A_3$ invertible?

(d) Let $A_4 = \begin{pmatrix} 3 & -1 & 5 \\ 2 & 1 & 3 \\ 0 & -5 & 1 \end{pmatrix}$. Find a basis for *Row* $A_4$. What is the rank of $A_4$? What is the dimension of $Nul\ A$?

(e) Let $A_5 = \begin{pmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & 2 \end{pmatrix}$. Find a basis for $Col\ A_5$. What is the rank of $A_5$?

(f) Let $A_6 = \begin{pmatrix} 13 & -5 & 2 \\ -5 & 13 & 2 \\ 2 & 2 & 5 \end{pmatrix}$. Compute the determinant of $A_6$. Is $A$ invertible?

(g) Let $A_7 = \begin{pmatrix} \frac{13}{6} & -\frac{5}{6} & \frac{1}{3} \\ -\frac{5}{6} & \frac{13}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{10}{6} \end{pmatrix}$. The eigenvalues of this matrix are $1, 2$ and $3$. Find all the eigenvectors of $A_7$. Is $A_7$ diagonalizable? If so, give the diagonalization.

(3) (10 points) Let $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) Compute the reduced singular value decomposition of $B$. Does $B$ have a trivial or non-trivial null space? What is the rank of $B$?

(b) Find the pseudo-inverse of $B$.

(4) (10 points) Let $Q$ be an $n \times n$ orthogonal matrix and $A$ an $n \times m$ matrix. Show that $A$ and $QA$ have the same singular values.

(5) (10 points) Let $C$ be a $3 \times 3$ symmetric matrix with orthogonal diagonalization given by $C = PDP^{-1}$ where the columns of $P$ are $\{\vec{p_1}, \ldots, \vec{p_n}\}$ and the nonzero entries of the matrix $D$ are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ where $r < n$. Let $\mathfrak{B}$ denote the basis of eigenvectors of $C$.

(a) What is the change of basis matrix from the standard basis to $\mathfrak{B}$? What is the change of basis matrix from $\mathfrak{B}$ to the standard basis (do not just state this as an inverse of another matrix)?

(b) What is $[C]_{\mathfrak{B}}$? Justify your answer.

(6) (5 points) Let

$$D = \begin{pmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$D$ has a $QR$ decomposition given by

$$D = QR = \begin{pmatrix} \frac{1}{2} & \frac{3\sqrt{5}}{10} & -\frac{\sqrt{6}}{6} \\ -\frac{1}{2} & \frac{3\sqrt{5}}{10} & 0 \\ -\frac{1}{2} & \frac{\sqrt{5}}{10} & \frac{\sqrt{6}}{6} \\ \frac{1}{2} & \frac{\sqrt{5}}{10} & \frac{\sqrt{6}}{3} \end{pmatrix} \begin{pmatrix} 2 & 1 & \frac{1}{2} \\ 0 & \sqrt{5} & \frac{3\sqrt{5}}{2} \\ 0 & 0 & \frac{\sqrt{6}}{2} \end{pmatrix}$$

Using the QR factorization, find the least squares solution to $A\vec{x} = \vec{b}$ where

$$\vec{b} = \begin{pmatrix} 2 \\ -3 \\ -2 \\ 0 \end{pmatrix}$$

(7) (10 points) Describe and explain the Gram-Schmidt algorithm.

(8) (10 points) Consider the following data series:

| | | | | | |
|---|---|---|---|---|---|
| x | 1 | 2 | 3 | 4 | 5 |
| y | 0 | 2 | 1 | 4 | 5 |

Suppose we wish to construct a general linear model of the form $y = \beta_1 x + \beta_2 x^3$. What is are design matrix, observation vector and parameter vector for this model? Write down the normal equations for this model but do not solve them.

(9) (10 points) Let $A$ be an $m \times n$ matrix. Show that $Nul\ A$ is a subspace of $\mathbb{R}^n$ and that $Row\ A$ is its orthogonal complement.

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