Your name:

Instructor (please circle): Samantha Allen Angelica Babei

Math 22 Fall 2018 Homework 8, due Fri Nov 9 4:00 pm in homework boxes in front of Kemeny 108 *Please show your work, and check your answers. No credit is given for solutions without work or justification.*

(1) Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$. Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.

(a) Find a vector $\mathbf{v}_3$ such that the set $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set.

Need $\mathbf{v}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $\mathbf{v}_3 \cdot \mathbf{v}_1 = 0$ and $\mathbf{v}_3 \cdot \mathbf{v}_2 = 0$. So solve

$$2a - b = 0,$$
$$2a + 4b - 3c = 0.$$

One correct answer would be

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 10/3 \end{bmatrix}.$$

(b) Normalize each vector in B to find an orthonormal basis B' for $\mathbb{R}^3$.

$$||\mathbf{v}_1|| = \sqrt{5},\ ||\mathbf{v}_2|| = \sqrt{29},\ ||\mathbf{v}_3|| = \sqrt{145}/3$$

$$B' = \left\{ \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 2/\sqrt{29} \\ 4/\sqrt{29} \\ -3/\sqrt{29} \end{bmatrix}, \begin{bmatrix} 3/\sqrt{145} \\ 6/\sqrt{145} \\ 10/\sqrt{145} \end{bmatrix} \right\}$$

(c) Write $\mathbf{y} = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}$ as a linear combination of the vectors in B'.

Say $B' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Then

$$\mathbf{y} = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\mathbf{u}_2 + (\mathbf{y} \cdot \mathbf{u}_3)\mathbf{u}_3$$
$$= (3/\sqrt{5})\mathbf{u}_1 - (18/\sqrt{29})\mathbf{u}_2 + (2/\sqrt{145})\mathbf{u}_3$$

(d) Find the distance from $\mathbf{y}$ to the subspace W of $\mathbb{R}^3$ spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$d = ||\mathbf{y} - \mathbf{proj_W y}|| = ||\mathbf{y} - ((3/\sqrt{5})\mathbf{u}_1 - (18/\sqrt{29})\mathbf{u}_2)||$$
$$= ||(2/\sqrt{145})\mathbf{u}_3|| = 2/\sqrt{145}.$$

(2) True or false (no working needed, just circle the answer):

(a) T : If A is a 6×5 matrix such that $\dim \mathrm{Col} A = 3$, then $\dim\left((\mathrm{Row} A)^{\perp}\right) = 2$.

(b) F: If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is an orthogonal set of vectors in $\mathbb{R}^n$, then S is a basis for $\mathbb{R}^n$.

(c) T : If U is a square matrix with orthonormal columns, then U is invertible.

(d) T : For any subspace W of $\mathbb{R}^n$, the only element which is in both W and $W^{\perp}$ is the zero vector.

(e) F : If two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthgonal, then $||\mathbf{u} + \mathbf{v}|| < ||\mathbf{u}|| + ||\mathbf{v}||$.

(3) Consider the Markov chain given by transition matrix $P = \begin{bmatrix} 0 & 0.2 \\ 1 & 0.8 \end{bmatrix}$ and initial vector $\mathbf{x}_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$.

(a) Show that P is a regular matrix.
First, note that the columns of P are probability vectors: all of the entries are nonnegative, and the columns add to one.
Second, we have that $P^2 = \begin{bmatrix} 0 & 0.2 \\ 1 & 0.8 \end{bmatrix} \begin{bmatrix} 0 & 0.2 \\ 1 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.16 \\ 0.8 & 0.84 \end{bmatrix}$. Since P^2 has nonzero entries, P is a regular matrix.

(b) Find $\mathbf{x}_2$.

$$\mathbf{x}_2 = P^2\mathbf{x}_0 = \begin{bmatrix} 0.2 & 0.16 \\ 0.8 & 0.84 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.18 \\ 0.82 \end{bmatrix}$$

(c) Find the steady-state vector $\mathbf{q}$ for P.
We need to find $\mathbf{q}$ such that $P\mathbf{q} = \mathbf{q}$ and $\mathbf{q}$ is a probability vector.
First, find an eigenvector of P corresponding to the eigenvalue 1.
So solve $(P - 1I_2)\mathbf{x} = \mathbf{0}$:

$$\left[\begin{array}{rr|r} -1 & 0.2 & 0 \\ 1 & -0.2 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} -1 & 0.2 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -0.2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}$$

So $\begin{bmatrix} 0.2 \\ 1 \end{bmatrix}$ is an eigenvector of P corresponding to the eigenvalue 1. Note that $0.2 + 1 = 1.2$ and let $\mathbf{q} = \frac{1}{1.2} \begin{bmatrix} 0.2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 5/6 \end{bmatrix}$.