Your name:

Instructor (please circle): Samantha Allen Angelica Babei

Math 22 Fall 2018 Homework 7, due Fri Nov 2 4:00 pm in homework boxes in front of Kemeny 108 *Please show your work, and check your answers. No credit is given for solutions without work or justification.*

(1) Consider

$$A = \begin{bmatrix} 2 & -2 & 1 \\ -1 & 3 & -1 \\ 2 & -4 & 3 \end{bmatrix}.$$

(a) What is the characteristic polynomial of A? Find all of the eigenvalues for A, and state their algebraic multiplicities.

Solution.

$$\det(A - \lambda I_3) = (2-\lambda)\det\begin{bmatrix} 3-\lambda & -1 \\ -4 & 3-\lambda \end{bmatrix} + 2\det\begin{bmatrix} -1 & -1 \\ 2 & 3-\lambda \end{bmatrix} + \det\begin{bmatrix} -1 & 3-\lambda \\ 2 & -4 \end{bmatrix}$$

$$= (2-\lambda)(3-\lambda)(3-\lambda) - 4(2-\lambda) + 2(-(3-\lambda)+2) + 4 - 2(3-\lambda) = -\lambda^3 + 8\lambda^2 - 13\lambda + 6$$

We want to solve $\lambda^3 - 8\lambda^2 + 13\lambda - 6 = 0$. The integer roots of a polynomial with integer entries are factors of its constant coefficient. In this case, we need to check $\pm1, \pm2, \pm3, \pm6$. We see that $\lambda = 1$ is a root. Using long division, we get that $\lambda^3 - 8\lambda^2 + 13\lambda - 6 = (\lambda-1)(\lambda^2 - 7\lambda + 6) = (\lambda-1)(\lambda-1)(\lambda-6)$, so we have eigenvalues $\lambda = 1$ with algebraic multiplicity 2, and $\lambda = 6$ with algebraic multiplicity 1.

(b) For each of the eigenvalues found in part (a), find the dimensions of their respective eigenspaces.

Solution. $\lambda = 1$, then

$$\begin{bmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ 2 & -4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so the eigenspace for $\lambda = 1$ has dimension 2, as there are two free variables.

$\lambda = 6$, then

$$\begin{bmatrix} -4 & -2 & 1 \\ -1 & -3 & -1 \\ 2 & -4 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so the eigenspace for $\lambda = 6$ has dimension 1, as there is one free variables.

(2) True or false (no working needed, just circle the answer):

(a) F: An $n \times n$ matrix A is invertible if and only if $\lambda = 0$ is an eigenvalue.

(b) T : If $\mathbf{x}$ is an eigenvector for A, so is $3\mathbf{x}$.

(c) F: Every matrix is similar to a diagonal matrix.

(d) T : If $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors for A in the same eigenspace, then any nonzero linear combination of $\mathbf{u}$ and $\mathbf{v}$ is also an eigenvector.

(e) F: A matrix is diagonalizable if and only if the algebraic multiplicity of each eigenvalue is equal to the dimension of its eigenspace.

(3) Consider

$$B = \begin{bmatrix} 17 & -6 \\ 45 & -16 \end{bmatrix}.$$

Find B^{11}. Explain all the intermediary steps, but no need to simplify the end result (for example, you may leave unsimplified entries in the matrices such as $8 \cdot 3^{12} - 4$).

Solution. We diagonalize. First we find the eigenvalues:

$$(17 - \lambda)(-16 - \lambda) + 270 = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2).$$

Since the charpoly factors completely into distinct linear factors, the matrix is diagonalizable.

$\lambda = -1$ gives

$$\begin{bmatrix} 18 & -6 \\ 45 & -15 \end{bmatrix} \to \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$$

so the eigenvector basis for the eigenspace for $\lambda = -1$ is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$\lambda = 2$ gives

$$\begin{bmatrix} 15 & -6 \\ 45 & -18 \end{bmatrix} \to \begin{bmatrix} 5 & -2 \\ 0 & 0 \end{bmatrix}$$

so the eigenvector basis for the eigenspace for $\lambda = 2$ is $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

Therefore, $B = PDP^{-1}$ for $P = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, and $P^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$.

Then

$$B^{11} = PD^{11}P^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2048 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}.$$