Your name:

Instructor (please circle): Samantha Allen Angelica Babei

Math 22 Fall 2018 Homework 3, due Fri Oct 5 4:00 pm in homework boxes in front of Kemeny 108 *Please show your work, and check your answers. No credit is given for solutions without work or justification.*

(1) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the function given by the formula

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 - x_3, 3x_1 + 6x_2, x_1 + 2x_2, x_2 + 4x_3)$$

(a) T is a linear transformation. What is the standard matrix A of T?

Solution.

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\3\\1\\0\end{bmatrix} \quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\6\\2\\1\end{bmatrix} \quad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\0\\0\\4\end{bmatrix} \quad \text{so } A = \begin{bmatrix}1 & 2 & -1\\3 & 6 & 0\\1 & 2 & 0\\0 & 1 & 4\end{bmatrix}.$$

(b) Is the transformation T one-to-one? Justify your answer.

Solution. A transformation T with standard matrix A is one-to-one if and only if its columns are linearly independent (or alternatively, if and only if the matrix equation $A\mathbf{x} = \mathbf{0}$ only has the trivial solution). This happens when the REF of A has a pivot in every column. To find out if this is the case, we row-reduce A to REF:

$$\begin{bmatrix}1 & 2 & -1\\3 & 6 & 0\\1 & 2 & 0\\0 & 1 & 4\end{bmatrix} \xrightarrow[R_2 \leftarrow R_2 - 3R_1]{R_3 \leftarrow R_3 - R_1} \begin{bmatrix}1 & 2 & -1\\0 & 0 & 3\\0 & 0 & 1\\0 & 1 & 4\end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_2} \begin{bmatrix}1 & 2 & -1\\0 & 1 & 4\\0 & 0 & 1\\0 & 0 & 3\end{bmatrix} \xrightarrow{R_4 \to R_4 - 3R_3} \begin{bmatrix}1 & 2 & -1\\0 & 1 & 4\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$

Therefore, T is indeed one-to-one.

(c) Is the transformation T onto? Justify your answer.

Solution. T is onto if and only if the columns of its standard matrix A span the codomain, which happens if and only if A has a pivot in each row. Since A has more rows than columns, we see that this is not the case, and T is not onto.

(2) True or false (no working needed, just circle the answer):

(a) T : Every matrix transformation is a linear transformation.

(b) F: The function $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by the formula

$$T(x_1, x_2) = (x_1 - x_2, 5x_1 + 2x_2, x_2 - 5)$$

is a linear transformation.

(c) T : If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation corresponding to counter-clockwise rotation of $3\pi/4$ (or 135°), then $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}.$$

(d) T : A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ is never onto.

(e) F: For any 3×3 matrices A and B, $AB = BA$.

(3) Let A, B, C be the following matrices:

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \\ 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 \\ -2 & 0 \end{bmatrix}$$

(a) Calculate AB, AC, and $AB + AC$.
Solution.

$$AB = \begin{bmatrix} 8 & 11 \\ -4 & 2 \\ 0 & 15 \end{bmatrix}, \quad AC = \begin{bmatrix} -2 & -2 \\ -4 & 1 \\ -10 & 0 \end{bmatrix}, \quad AB + AC = \begin{bmatrix} 6 & 9 \\ -8 & 3 \\ -10 & 15 \end{bmatrix}$$

(b) Calculate $B + C$ and then multiply A and $B + C$ to get $A(B + C)$.

$$B + C = \begin{bmatrix} 6 & 0 \\ -2 & 3 \end{bmatrix}, \quad A(B + C) = \begin{bmatrix} 6 & 9 \\ -8 & 3 \\ -10 & 15 \end{bmatrix}$$