Your name:

Instructor (please circle): Alex Barnett Michael Musty

Math 22 Summer 2017, Midterm 1, Wed July 19

Please show your work. No credit is given for solutions without work or justification.

1. [8 points] Consider the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 2 & 4 & 1 & 1 \\ -1 & -2 & 0 & -1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

(a) Is the system consistent? If not, explain why. If consistent, write the general solution in *parametric vector form*:

(b) Express the solution set to the corresponding *homogeneous* system $A\mathbf{x} = \mathbf{0}$ in the form of a span of one or more vectors:

2. [9 points]

(a) Find the inverse of $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 7 \\ -1 & 1 & -3 \end{bmatrix}$, if it exists, or prove that it does not exist:

(b) Using just the definition of invertibility, prove that if A is invertible, the linear system $A\mathbf{x} = \mathbf{b}$ is consistent for all right-hand sides. [Note: this is one of the parts of the Invertible Matrix Theorem, so you cannot use the IMT in your proof!]

(c) Prove that if A^2 is invertible, then A is too.

3. [7 points]

(a) Determine the value(s) of x so that the vectors $\begin{bmatrix} 1 \\ x \end{bmatrix}$ and $\begin{bmatrix} x \\ x+2 \end{bmatrix}$ are linearly independent:

(b) Find the standard matrix for the linear transformation from the plane to itself which rotates all points clockwise by $\pi/2$:

4. [9 points] Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ with standard matrix

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -4 & 0 \\ 3 & -5 & -10 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix}.$$

(a) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a set of vectors with each vector in $\mathbb{R}^n$. State the precise definition of what it means for this set of vectors to be linearly independent.

(b) Is the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ (consisting of the columns of A) linearly independent? If so, then prove your answer. If not, provide an explicit dependence relation.

(c) Is T one-to-one? (yes or no will suffice)

(d) Is T onto? Explain why or why not.

5. [7 points] Consider a human and zombie population. Each year a quarter of the humans become zombies and half the zombies die by various means. Let

$$\mathbf{x}_k = \begin{bmatrix} h_k \\ z_k \end{bmatrix}$$

be the state of the system k years after the zombie outbreak (where h_k is the human population k years after the zombie outbreak, and z_k is the zombie population k years after the zombie outbreak).

(a) Find a 2×2 migration matrix A such that $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

(b) Suppose $h_1 = 500$ and $z_1 = 275$. Use A^{-1} to find h_0 and z_0.

BONUS Find the equilibrium vector $\mathbf{x}$ that is unchanged by multiplication by A.

6. [10 points] In this question only, no working is needed; just circle T or F.

(a) T / F: The following is a valid proof that if A is invertible, so is A^T. The REF of A has a pivot in every row, so the REF of A^T would have a pivot in every column, so A^T row reduces to I, so A^T is invertible.

(b) T / F: The following is a valid proof that if A is invertible, its columns are linearly independent. Let $\mathbf{x}$ solve $A\mathbf{x} = \mathbf{0}$. Left-multiply by A^{-1} to get $\mathbf{x} = \mathbf{0}$. Thus the columns of A are linearly indepedent.

(c) T / F: For any $n \times n$ matrix, the transpose of the inverse is the inverse of the transpose.

(d) T / F: A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ being one-to-one means that T maps every $\mathbf{x}$ in $\mathbb{R}^n$ to a unique vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

(e) T / F: If the matrix equation $A\mathbf{x} = \mathbf{b}$ is inconsistent, then $\mathbf{b}$ cannot be written as a linear combination of the columns of A.

(f) T / F: If the matrix equation $A\mathbf{x} = \mathbf{0}$ has a solution, then there is a dependence relation among the columns of A.

(g) T / F: $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 1 \\ x_1 + 1 \\ x_1 + x_2 \end{bmatrix}$ is linear.

(h) T / F: It is possible to define a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$ that is one-to-one.

(i) T / F: If the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is linearly dependent, then $\mathbf{a}_1$ is in $\text{Span}\{\mathbf{a}_2, \mathbf{a}_3\}$.

(j) T / F: If A is a square matrix, and the matrix equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A is invertible.