CHANGE OF BASIS WORKSHEET

OCTOBER 20, 2017

Let V be an n-dimensional vector space and $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be bases for V. The following exercises will walk you through a proof of Theorem 15.

(1) (a) Given $\mathbf{v} \in V$, why can we find scalars $x_1, \ldots, x_n$ such that

$$\mathbf{v} = x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n\,?$$

(b) Apply the coordinate mapping $\varphi_{\mathcal{C}} = [\ \ ]_{\mathcal{C}}$ to both sides of the above equation. What does this yield? What property of $\varphi_{\mathcal{C}}$ are you using?

(c) Rewrite your answer from the previous part as a matrix product involving ${}_{\mathcal{C}}[\mathrm{id}]_{\mathcal{B}}$.

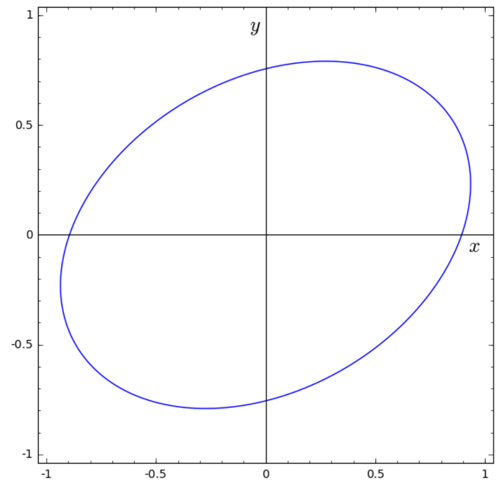
(d) Explain how the above steps prove that ${}_{\mathcal{C}}[\mathrm{id}]_{\mathcal{B}}$ satisfies the property stated in Theorem 15.

(2) Suppose Q is a matrix such that $[\mathbf{v}]_{\mathcal{C}} = Q[\mathbf{v}]_{\mathcal{B}}$ for all $\mathbf{v} \in V$.

(a) Substitute $\mathbf{v} = \mathbf{b}_1$ in the above equation. What does this tell you about the first column of Q? Similarly, what does substituting $\mathbf{b}_2, \ldots, \mathbf{b}_n$ for $\mathbf{v}$ yield?

(b) Why does this prove the uniqueness statement in Theorem 15?

(3) Consider the ellipse E given by the equation $\frac{5}{4}x^2 - \frac{\sqrt{3}}{2}xy + \frac{7}{4}y^2 = 1$, where the points $\begin{pmatrix} x \\ y \end{pmatrix} = x\mathbf{e}_1 + y\mathbf{e}_2$ are written with respect to the standard basis for $\mathbb{R}^2$. The major and minor axes of E form an angle of 30° with the coordinate axes.



In this problem, you will find a change of basis that yields a simpler equation for E.

(a) Let $\mathcal{B}$ be the basis consisting of unit vectors parallel to the major and minor axes of the ellipse. Compute the change of basis matrix ${}_{\mathcal{E}}[\mathrm{id}]_{\mathcal{B}}$.

(b) Given a point $\mathbf{p} \in \mathbb{R}^2$, write $\begin{pmatrix} u \\ v \end{pmatrix}$ for the $\mathcal{B}$-coordinate vector of $\mathbf{p}$. Use ${}_{\mathcal{E}}[\mathrm{id}]_{\mathcal{B}}$ to express x and y in terms of u and v.

(c) Substitute the expressions for x and y found in the previous part into the equation for E. What is the resulting equation for E with respect to these new coordinates?