Methods of proof.

Math 22, Summer 2017, Alex Barnett, X-hour 6/29/17

Let's say we have some facts and definitions "on the table": you are a human, a human is a mammal, and a mammal is a warm-blooded animal. You are only allowed to build new results from these ones.

- "By example:" eg. **Theorem 1**. Mammals exist.

  Proof. I am a human, so at least one human exists. Humans are mammals. □

  Comment: Finding such an example is a *creative act*. In this simple setting you didn't have to look very far.

- "Direct proof:" (apply definitions & use logic). eg. **Theorem 2**. Human x is warm-blooded.

  Proof. Humans are mammals. Mammals are warm-blooded. □

  Comment: here the creative act is finding a *chain* of logic that connect the two concepts in the statement ("human" and "warm-blooded").

- "By contradiction:" (assume the opposite of the claim then from it derive something impossible). Eg. prove Theorem 2 a different way:

  Proof. Assume x is cold-blooded. It follows that a mammal would be cold-blooded, which contradicts the definition of mammal. □

- "By contrapositive:" eg. **Theorem 3**. Let x be an animal and let x be cold-blooded. Then x is not a human.

  The form is $A \Rightarrow B$. We use $\neg A$ to mean "not A", ie the statement A doesn't hold. Then $A \Rightarrow B$ is equivalent to $\neg B \Rightarrow \neg A$, which is called the *contrapositive*. Why equivalent? Check the 2×2 matrix of possibilities of A being false or true, and B being false or true. The theorem and its contrapositive both exclude "A and $\neg B$" but allow the other three possibilities.

  Let's use it.

  Proof. $\neg B$ is "x is a human". $\neg A$ is "x is warm-blooded." But the implication "x is a human $\Rightarrow$ x is warm-blooded" is precisely Theorem 2 above, which we already proved. □

Exercises (relevant to Sec 1.4, 1.5): Prove, using M22 facts so far, and state the proof type:

1. Let A be a square matrix. If $A\mathbf{x} = \mathbf{b}$ is consistent for all RHS (right hand sides) $\mathbf{b}$, then the solution is always unique.

2. Let A be any $m \times n$ matrix. Then the linear system $A\mathbf{x} = \mathbf{0}$ is consistent.

3. Let $\text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\} = \mathbb{R}^m$, then $n \geq m$. (A more chatty version is: "You need at least m vectors to span $\mathbb{R}^m$". But note you have to handle the more dry language too.)

For more detail see documents on the Resources page. Also see our other proof worksheets.

Solutions to exercises:

1. "Direct" proof: Consistent for all RHS $\Rightarrow$ pivot in every row (from lecture). Square and pivot in every row $\Rightarrow$ pivot in every column. Combining the last two, we have there is a pivot in every column. This implies uniqueness, whatever the RHS actually is. $\square$

   There is also a "contrapositive" version which is the negation of each step in reverse order.

2. Proof "by example": $\mathbf{x} = \mathbf{0}$ is a solution. $\square$

3. Proof "by contradiction": Suppose $n < m$ vectors did span $\mathbb{R}^m$, then there would be a pivot in every row, thus at least m pivots. But there can be at most one pivot in each of the n columns. Contradiction. $\square$