Your name: $\sim$ SOLUTIONS $\sim$

Instructor (please circle): Alex Barnett Naomi Tanabe

Math 22 Fall 2016, Final, Fri Nov 18 (80 points total)

Please show your work. No credit is given for solutions without work or justification.

1. [10 points]

3pts (a) Is the matrix $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ diagonalizable?

find roots of $\det(A-\lambda I)$:

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2 = 0$$

Eigenvalues $\lambda = 2$ (twice)

$$A - 2I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \overset{\text{reduce}}{\sim} \begin{bmatrix} \boxed{1} & 1 \\ 0 & 0 \end{bmatrix}$$

x_2 free

dim Nul $(A-2I) = 1$, not full set of eigenvectors. $\Rightarrow$ no.

2pts (b) Is the matrix $\begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$ diagonalizable?

$$\begin{vmatrix} 4-\lambda & -2 \\ 3 & -1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1) = 0$$

$\Rightarrow \lambda = 2, 1$ are distinct, so yes, diagonalizable

2pts (c) Let A be whichever (b) of the above matrices was diagonalizable. Give a matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

$\lambda_1 = 1$: $A - 1I = \begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix}$ so $\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\lambda_2 = 2$: $A - 2I = \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$ so $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\Rightarrow D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad P = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$

3 pts. (d) Let A be the same matrix (ie the diagonalizable one, $A = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$) as in (c). Give an expression (involving only numbers) for the result when the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is multiplied by A from the left 2016 times.

We need $P^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$

Then answer $\vec{y} = A^k \vec{x}$, where $k = 2016$, $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$= PD^kP^{-1}\vec{x} \qquad \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= P\begin{bmatrix} 1^k & 0 \\ 0 & 2^k \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

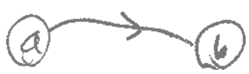
or use $c_1\lambda_1^k\vec{v_1} + c_2\lambda_2^k\vec{v_2}$

for all points should simplify to a single vector or sum of two vectors.

$$= \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \cdot 2^k \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} + (3)2^{2016}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 + (3)2^{2016} \\ -3 + (3)2^{2016} \end{bmatrix}$$

2. [11 points]

5 pts. (a) Suppose the web consists of two pages: page a links to page b (that's it!) Use the PageRank algorithm (with $\alpha = 1$) to compute the vector $\mathbf{q}$ of importance scores. Scale your answer so that it is a probability vector:



Build stochastic matrix
$$A = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix} \end{array}$$

2 pts

The 2nd column is all $1/n$ entries ($n=2$) since page b has no outgoing links. This is equivalent to jumping to a random page.

Solve $A\vec{q} = \vec{q}$, or find a null vector of $A - I$:

$$A - I = \begin{bmatrix} -1 & 1/2 \\ 1 & -1/2 \end{bmatrix} \overset{\text{reduce}}{\sim} \begin{bmatrix} \boxed{1} & -1/2 \\ 0 & 0 \end{bmatrix}$$

x_2 free

so $x_1 = +\frac{1}{2}x_2$
$x_2 = x_2$

$$\vec{x} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}x_2 \xrightarrow[\text{(probability vector)}]{\text{normalize so entries sum to 1}} \vec{q} = \frac{1}{3/2}\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

2 pts

1 pt for normalize

reality check: b higher score than a.

2pts (b) As you know, in the real world the number n of web pages exceeds 10^{10}, making the above linear solve impossibly expensive. Explain in one sentence how Google in practice approximates $\mathbf{q}$.

ie $x \leftarrow Ax$

They pick an arbitrary starting guess $\vec{x}^{(0)}$ then evolve the Markov chain $\vec{x}^{(k+1)} = A\vec{x}^{(k)}$ by doing "mat-vecs" until it has converged to something close to $\vec{q}$.

[Partial credit also for mentioning purpose of setting $\alpha \approx 0.85$, to ensure regular.]

2pts (c) Give the definition of a $n \times n$ matrix being *stochastic*.

i) A should have all entries non-negative

ii) Each column of A should sum to 1.

2pts (d) Prove that any $n \times n$ stochastic matrix A has an eigenvalue of 1.

*note, assuming $A\vec{q} = \vec{q}$ for some $\vec{q}$ is not a valid starting point, since this obviously means 1 is eigval

Several ways to prove this:

i) $A - I$ must have columns summing to 0 since A has columns summing to 1. Thus there is a nontrivial linear combination of rows which vanishes, so the rows are not lin. indep, so rank$(A-I) < n$, ie $\dim \operatorname{Nul}(A-I) > 0$, so $\lambda = 1$ is an eigenvalue.

OR ii) $A^T\vec{v} = \vec{v}$, where $\vec{v} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, so A^T has an eigenvalue 1, so A does too.

BONUS Prove that the procedure in (b) always works for your matrix from (a). Estimate how long it would take to get to 1% accuracy in $\mathbf{q}$.

+1pt { For any starting probability vector $\vec{x}^{(0)}$, Markov chain converges to unique $\vec{q}$ if can show either

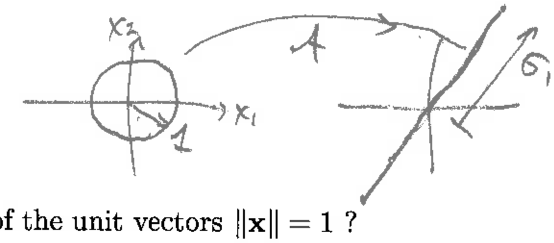
i) A is regular : true since $A^2 = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}$, all entries > 0

or ii) $\lambda_1 = 1$ & all other $|\lambda_j| < 1$. True since $\lambda_1 = 1$, $\lambda_2 = -1/2$.

+1pt. { For convergence rate, since $\vec{x}^{(k)} = \underbrace{c_1\lambda_1^k\vec{v}_1}_{\text{the answer } \vec{q}} + \underbrace{c_2\lambda_2^k\vec{v}_2}_{\text{error term}}$

Since $|\lambda_2| = 1/2$, error term $\approx 1/2^k$

Setting $2^{-k} \approx 0.01$ gives $k \approx 7$ iterations enough.



3. [10 points] Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

4pts (a) What is the maximum $\|A\mathbf{x}\|$ can have over all of the unit vectors $\|\mathbf{x}\| = 1$?

I.e., what is σ_1, the largest (first) singular vector?

$\sigma_1 = \sqrt{\lambda_1(A^TA)}$ ← biggest. $\quad A^TA = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$

Eigenvalues: $\begin{vmatrix} 5-\lambda & 5 \\ 5 & 5-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + \cancel{5^2} - \cancel{5^2} = 0$, $\lambda = 0, 10$

so $\sigma_1 = \sqrt{10}$

5pts (b) Compute the *full* singular value decomposition of A (i.e. give U, Σ and V. Choose signs so that the top row of V has positive entries.)

$\vec{v}_j$ = Eigenvectors of A^TA: $\quad \lambda_1 = 10$: $A^TA - 10I = \begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix}$

$\lambda_2 = 0$, so $\sigma_2 = 0$. $\quad \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ normalize.

Since A^TA symmetric we know $\vec{v}_2$ orthog. to $\vec{v}_1$, so $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \xrightarrow{\text{norm.}} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$\vec{u}_1$ = normalized $A\vec{v}_1 = \dfrac{1}{\sqrt{2^2+4^2}}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

Can't get $\vec{u}_2$ from $A\vec{v}_2$ since it's $\vec{0}$. But $\vec{u}_2$ must complete an o.n.b. for $\mathbb{R}^2$, so be orthog. to $\vec{u}_1$. So, $\vec{u}_2 = \pm\begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$

Thus $A = U\Sigma V^T$ where

$$U = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

2nd col. of U can have either overall sign.

1pt (c) Which row(s) or column(s) of which matrix gives an orthonormal basis for Nul (A^T) ?

$\{\vec{u}_2\}$ is o.n.b. for $(\text{Col} A)^\perp = \text{Nul}(A^T)$, since rank $A = 1$

4. [10 points] Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$ and the vector $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$.

2pt. (a) Is $\mathbf{b}$ in Col A ?

Check consistency of lin. sys.

aug. matrix: $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & -5 \end{array}\right] \overset{\text{reduce}}{\sim} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & -2 & 2 \\ 0 & 1 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 12 \end{bmatrix} \leftarrow$ incons. !

$\Rightarrow$ no

4pts. (b) Find the complete set of least squares solutions to $A\mathbf{x} = \mathbf{b}$.

Use normal equations: $A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & -1 \\ 2 & -1 & 5 \end{bmatrix}$ ← symm.

$A^Tb = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

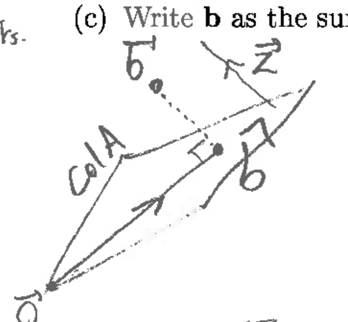
Solve $A^TA\hat{x} = A^Tb$: $\left[\begin{array}{ccc|c} 2 & 2 & 2 & 2 \\ 2 & 5 & -1 & -1 \\ 2 & -1 & 5 & 5 \end{array}\right] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & -3 & -3 \\ 0 & -3 & 3 & 3 \end{bmatrix}$

REF $\sim \begin{bmatrix} \boxed{1} & 0 & 2 & 2 \\ 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ← free. x_3

$\Rightarrow$ $x_1 = 2 - 2x_3$
$x_2 = -1 + x_3$
$x_3 = x_3$

ie $\hat{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} t$, t real. ← Full solution set.

2pts. (c) Write $\mathbf{b}$ as the sum of some vector in Col A and some vector orthogonal to Col A.



$\hat{b} = A\hat{x} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}_{=\vec{0}} t$

$= \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$

$= \vec{0}$, as know since Nullspace of normal eqns same as that of A itself.

Thus $\vec{z} = \vec{b} - \hat{b} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}$

$\vec{b} = \underset{\hat{b} \text{ in Col } A}{\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}} + \underset{\vec{z} \text{ in } (\text{Col } A)^{\perp}}{\begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}}$

2pt. (d) Is your answer to (c) unique? Why? Yes, either because of the way the nonuniqueness in $\hat{x}$ fell in Nul A in (c), or, better... the Orthogonal Decomposition Theorem says any such decomp. into W and $W^\perp$ is unique.

5. [9 points] Consider the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 4 \end{bmatrix}$.

$\vec{x}_1$ $\vec{x}_2$

4pts (a) Find an orthogonal basis for Col A. ← Use Gram-Schmidt, since you see QR coming up in (b) anyway...

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} - \frac{(-4)}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$\{\vec{v}_1, \vec{v}_2\}$ are orthog. basis.

3pts. (b) Compute the QR decomposition of A.

Normalize each basis element: $\vec{v}_1 \to \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$, $\vec{v}_2 \to \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$

Stack into $Q = [\vec{v}_1 \; \vec{v}_2] = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 0 & 1/3 \\ -1/\sqrt{2} & 2/3 \end{bmatrix}$

$$R = Q^TA = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 2/3 & 1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -2\sqrt{2} \\ 0 & 3 \end{bmatrix}$$

← check it's upper-tri. ✓

2pts. (c) Use your previous answer to write a formula for the matrix that maps any point **b** in $\mathbb{R}^3$ to its *nearest* point in Col A. [You can leave it as an expression; don't write out matrix elements ... unless you enjoy pain]

$$\hat{b} = \underbrace{QQ^T}\vec{b}$$

this matrix is the orthogonal projector. By the Best Approximation Theorem the orthog. projection is the closest point in the subspace.

*Note if A full rank, then $\hat{x} = R^{-1}Q^T\vec{b}$, but $\hat{x}$ not asked for in (c)

6. [10 points] Short ones.

4 pts. (a) Let $A = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & a \\ a & 2 & 3 \end{bmatrix}$ with a real number a. For what values of a is the matrix A invertible?

Let's try to row reduce to I_3:

$$A \underset{R_3 \leftarrow R_3 - aR_1}{\sim} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & a \\ 0 & 2 & 3-a^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & a \\ 0 & 0 & 3-a^2-2a \end{bmatrix}$$

if vanishes, not invertible.

$a^2+2a-3 = (a+3)(a-1) = 0$ so $a = 1, -3$ not invertible.

Invertible for $\{a \text{ in } \mathbb{R},\ a \neq 1,\ a \neq -3\}$

3pts (b) Suppose that a general matrix A has linearly independent columns. Prove that PA, for any invertible matrix P, also has linearly independent columns. [Hint: if stuck first prove the converse.]

only get 1/3 if assumed A square!

Suppose $\vec{x}$ satisfies $PA\vec{x} = \vec{0}$. If we can prove from this that $\vec{x} = \vec{0}$, we're done. Left-mult. by P^{-1} which exists since P invertible

$$\underbrace{P^{-1}P}_{I} A\vec{x} = \vec{0} \quad \Rightarrow \quad A\vec{x} = \vec{0}$$

But we're given A has L.I. columns, thus $\vec{x} = \vec{0}$. ☒

[The converse is simpler: $A\vec{x} = \vec{0} \overset{\text{left mult.}}{\Rightarrow} PA\vec{x} = \vec{0} \overset{\text{L.I.}}{\Rightarrow} \vec{x} = \vec{0}$, partial credit.]

3pts (c) A matrix A such that $A^TA = AA^T$ is called "normal". Prove that for any normal matrix and any $\mathbf{x}$, the length of $A\mathbf{x}$ equals the length of $A^T\mathbf{x}$.

'length' = Euclidean length = 2-norm.

*note: normal ⇏ symmetric, amazingly.

$$\|Ax\|^2 = (Ax)\cdot Ax = x^T(A^TA)x = x^T(AA^T)x = \|A^T\vec{x}\|^2$$

Now, Square root both sides: $\|Ax\| = \|A^Tx\|$.

+1 pts. BONUS Prove the amazing fact that, for any square matrix, the sum of its eigenvalues equals the sum of its diagonal entries (its "trace"). You will need to use the back of the page.

Take $\det(A - \lambda I_n) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$ and match the coefficients of $(-\lambda)^{n-1}$.

The cofactor expansion on LHS gives $\prod_{j=1}^{n}(a_{jj}-\lambda)$ hence $\sum_{j=1}^{n} a_{jj}$; RHS gives $\sum_{j=1}^{n}\lambda_j$. ☒

7. [10 points] Shorter ones.

3pts (a) Let $W = \left\{ \begin{bmatrix} a \\ a \end{bmatrix}, a \text{ in } \mathbb{R} \right\}$. Prove whether or not W is a vector space.

Simplest way is: $W = \text{Span} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and any span is a subspace, hence a vector space in its own right.

OR, could check closed under addition, scalar mult., $\vec{0}$ is in W.

3pts (b) With W as above, every point in W is in the span of $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Is the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for W? Explain.

No, since neither $\vec{v}_1$ nor $\vec{v}_2$ is <u>in</u> W.

OR, $\text{Span}\{\vec{v}_1, \vec{v}_2\} = \mathbb{R}^2$ contains W but is not <u>equal</u> to W. (it's too 'big')

This is despite the fact that $\{\vec{v}_1, \vec{v}_2\}$ is Linearly indep. set.

4pts (c) Is $\left\{ \underset{\uparrow \vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\uparrow \vec{v}_2}{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}} \right\}$ a basis for the set $W = \text{Span} \left\{ \underset{\uparrow \vec{w}_1}{\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}}, \underset{\uparrow \vec{w}_2}{\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}} \right\}$? Explain. <u>Yes</u>.

Check two properties of a basis: i) $\{\vec{v}_1, \vec{v}_2\}$ is a linearly independent set, since $\vec{v}_2$ is not a multiple of $\vec{v}_1$.

& ii) $\{\vec{v}_1, \vec{v}_2\}$ are a spanning set for W. This must be checked: is each $\vec{w}_j$ in $\text{Span}\{\vec{v}_1, \vec{v}_2\}$?

We can test both in same aug. matrix:

$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & -2 & 2 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & 1 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \leftarrow \text{consistent for both } \vec{w}_1, \vec{w}_2.$$

Alternative for ii): argue $\dim W \leq 2$ & check both $\vec{v}_1, \vec{v}_2$ are <u>in</u> W (requires row reduction), & L.I., $\Rightarrow$ spanning set.

8. [10 points] In this question only, no working is needed; just circle T or F.

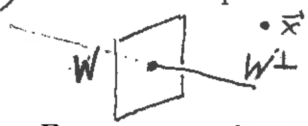
(a) T / F: The subset of $\mathbb{P}_2$ defined by $V = \{at^2 + bt + c \ : \ a, b, c \text{ real}, a \neq 0\}$, ie polynomials of degree precisely 2, is a subspace of $\mathbb{P}_2$.

the zero element $0t^2 + 0t + 0 \equiv 0$ is <u>not</u> in V.

(b) T / F: A 3×5 linear system can have a unique solution.

rank ≤ 3 so $\dim \operatorname{Nul} A = 5 - \operatorname{rank} A \geq 2$.

(c) T / F: If W is a subspace of $\mathbb{R}^n$, then every point in $\mathbb{R}^n$ is either in W or in $W^\perp$.



every $\vec{x}$ in $\mathbb{R}^n$ is <u>sum</u> of $\hat{x}$ in W & $\vec{z}$ in $W^\perp$

(d) T / F: Every symmetric $n \times n$ matrix has a set of eigenvectors which may be chosen to be an orthonormal basis for $\mathbb{R}^n$.

Orthogonal diagonalization theorem, §7.1.

(e) T / F: Every square matrix which is diagonalizable is invertible.

viz. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ These are independent properties.

(f) T / F: If A is an orthogonal square matrix, A^3 must also be orthogonal.

$$(A^3)^T A^3 = A^T A^T \underbrace{\cancel{A^T} \cancel{A}}_{I} A A = A^T \cancel{A^T} \cancel{A} A = A^T A = I$$

(g) T / F: The set of all unit vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

$\vec{0}$ is not in the set, etc.

(h) T / F: The set of 3-component vectors whose entries sum to zero is isomorphic to $\mathbb{R}^2$.

$W = \operatorname{Nul} [1 \ 1 \ 1]$ so $\dim W = 2$, isom. to $\mathbb{R}^2$.

(i) T / F: If $A = QR$ is the QR decomposition of a square matrix, then the eigenvalues of RQ are identical to those of A.

Watch this: $QRQ = AQ$ so $RQ = Q^T A Q$, similarity transform of A, same λ's.

"basis of the "QR algorithm for eigenvalues"."

(j) T / F: For any $m \times n$ matrix, rank A^T = rank A.

True since $\dim \operatorname{Col} A = \dim \underbrace{\operatorname{Row} A}_{\operatorname{Col}(A^T)}$

Rank theorem.