Diagonalization

11/11/03.
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For A), B), C) write (or try to write) the matrix in the form PDP^{-1} (P: real, invertible; D: diagonal) (be sure to state eigenvalue multiplicities!)

A) $\begin{bmatrix} -5 & 2 \\ -12 & 5 \end{bmatrix}$

$\lambda =$

$D = \begin{bmatrix} & \\ & \end{bmatrix}$

$P = \begin{bmatrix} & \\ & \end{bmatrix}$

B) $\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$

$\lambda =$

etc.

C) $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$\lambda =$

etc.

D) Write an expression for the matrix from A) to the power k ($k =$ integer)

[Hint: use its diagonal representation]

$$\begin{bmatrix} -5 & 2 \\ -12 & 5 \end{bmatrix}^k = \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} & \\ & \end{bmatrix}^k\begin{bmatrix} & \\ & \end{bmatrix} =$$

E) For $A = \begin{bmatrix} -5 & 2 \\ -12 & 5 \end{bmatrix}$, exploit $A^k = PD^kP^{-1}$ to evaluate $A^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$:

Diagonalization — SOLUTIONS

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For A), B), C) write (or try to write) the matrix in the form PDP^{-1} (be sure to state eigenvalue multiplicities!)

P: real, invertible; D: diagonal

A) $\begin{bmatrix}-5 & 2\\ -12 & 5\end{bmatrix}$ $\begin{vmatrix}-5-\lambda & 2\\ -12 & 5-\lambda\end{vmatrix} = (-5-\lambda)(5-\lambda)+24$

$= \lambda^2 - 1 = 0$ so $\lambda = +1, -1$

both multiplicity one.

$\lambda = 1, -1$

$D = \begin{bmatrix}1 & \\ & -1\end{bmatrix}$ ← note: order of columns has to match

$P = \begin{bmatrix}1 & 1\\ 3 & 2\end{bmatrix}$

$\vec{v}_1$, $\vec{v}_2$ (no space to show)

$\lambda_1 = +1$: $A - \lambda_1 I = \begin{bmatrix}-6 & 2\\ -12 & 4\end{bmatrix}$

so $\vec{v}_1 = \begin{bmatrix}1\\3\end{bmatrix}$ ← best choose integer entries.

B) $\begin{bmatrix}4 & -1\\ 1 & 2\end{bmatrix}$ $(4-\lambda)(2-\lambda)+1 = \lambda^2 - 6\lambda + 9$

$= (\lambda-3)^2 = 0$ so $\lambda = 3$, multiplicity two

$\lambda =$ etc. } not diagonalizable! ← so

$A - 3I = \begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix} \sim \begin{bmatrix}\boxed{1} & -1\\ 0 & 0\end{bmatrix}$ so only 1-dimensional eigenspace, not full set of $n=2$ eigenvectors.

free var.

C) $\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}$ $\lambda = 5$ (twice) & $A - 5I = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ 2 free vars.

so eg $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ are a basis for the eigenspace, which is all of $\mathbb{R}^2$.

$\lambda =$ etc. $\begin{bmatrix}5 & \\ & 5\end{bmatrix}$

$P = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ so $PDP^{-1} = D$, trivial case!

D) Write an expression for the matrix from A) to the power k (k = integer) [Hint: use its diagonal representation]

First, $P^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix} = \begin{bmatrix}-2 & 1\\ 3 & -1\end{bmatrix}$

$$A^k = \begin{bmatrix}-5 & 2\\ -12 & 5\end{bmatrix}^k = \underset{P}{\begin{bmatrix}1 & 1\\ 3 & 2\end{bmatrix}}\underset{D}{\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}^k}\underset{P^{-1}}{\begin{bmatrix}-2 & 1\\ 3 & -1\end{bmatrix}} = \begin{bmatrix}1 & 1\\ 3 & 2\end{bmatrix}\begin{bmatrix}-2 & 1\\ 3(-1)^k & -(-1)^k\end{bmatrix} = \begin{bmatrix}-2+3(-1)^k & 1-(-1)^k\\ -6+6(-1)^k & 3-2(-1)^k\end{bmatrix}$$

explicit formula for A^k

E) For $A = \begin{bmatrix}-5 & 2\\ -12 & 5\end{bmatrix}$, exploit $A^k = PD^kP^{-1}$ to evaluate $A^k\begin{bmatrix}1\\1\end{bmatrix}$:

either hit A^k above on $\begin{bmatrix}1\\1\end{bmatrix}$, or evaluate from scratch:

$$A^k\begin{bmatrix}1\\1\end{bmatrix} = \underset{P}{\begin{bmatrix}1 & 1\\ 3 & 2\end{bmatrix}}\underset{D^k}{\begin{bmatrix}1^k & 0\\ 0 & (-1)^k\end{bmatrix}}\underbrace{\underset{P^{-1}}{\begin{bmatrix}-2 & 1\\ 3 & -1\end{bmatrix}}\begin{bmatrix}1\\1\end{bmatrix}}_{\begin{bmatrix}-1\\2\end{bmatrix}\text{ coord vec in eigvecbasis.}} = \begin{bmatrix}1 & 1\\ 3 & 2\end{bmatrix}\begin{bmatrix}-1\\ 2(-1)^k\end{bmatrix} = \begin{bmatrix}-1+2(-1)^k\\ -3+4(-1)^k\end{bmatrix}$$

explicit dynamics!

Note, dynamics just flips one coordinate every timestep, other unchanged.