Your name:

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Math 22 Fall 2016, Homework 7, due Wed Nov 2

Please show your work. No credit is given for solutions without work or justification.

(1) Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, which you should check maps the vector $\mathbf{x}^{(k)} := \begin{bmatrix} f_k \\ f_{k-1} \end{bmatrix}$ to $\mathbf{x}^{(k+1)} := \begin{bmatrix} f_{k+1} \\ f_k \end{bmatrix}$, where $\{f_k\}_{k=0}^{\infty}$ is the Fibonacci sequence 1,1,2,3,5,8,. . . .

(a) Write the matrix in the form $A = PDP^{-1}$ (ie give all three matrices). You (and us) will find it easiest if you express answers using the *golden ratio* $\phi = (1+\sqrt{5})/2$; note $-\phi^{-1} = (1-\sqrt{5})/2$.

(b) Use this to write a formula for $\mathbf{x}^{(k)}$, for the dynamical system with matrix A, for the initial vector $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Reading off the first component gives an explicit formula for the kth Fibonacci number! (Isn't it weird that f_k must always be an integer?)

(c) To what does f_{k+1}/f_k converge, if anything, as $k \to \infty$, and why? (explain using your formula).

(2) You play the following "game." You start in the library. Every hour you toss a coin. If in the library, heads you stay in the library, tails you go party. If you're partying you still toss coins every hour but continue to party!

(a) Let the ordering be $\mathbf{x} = \begin{bmatrix} \text{probability of being in the library} \\ \text{probability of partying} \end{bmatrix}$. Write a stochastic matrix A whose Markov chain models the game.

(b) Derive a formula for the matrix A^k.

(c) Is the matrix A regular? (no explanation needed)

(d) After k hours (coin tosses), what is your probability vector $\mathbf{x}^{(k)}$?

(e) For any initial probability vector $\mathbf{x}^{(0)}$, does $\mathbf{x}^{(k)}$ converge to a unique steady state as $k \to \infty$, and why?

(f) Does (e) together with (c) contradict the convergence theorem for Markov chains? Explain.

(3) The web consists of three pages. a lints to b. b links to c. c links to a and b. Use the PageRank algorithm with $\alpha = 1$.

(a) Write the stochastic matrix then solve for the vector of importances (order it abc and normalize so that the vector sums to one).

(b) In a rather selfish effort to increase its rank (importance score), page a removes its link to b. [Hint: make sure you understand how the algorithm deals with such a situation.] What is the new vector of importances? Did the effort work?

BONUS If the PageRank Markov chain iteration were used to approximate solutions to (a) and (b), would they converge, and why?