PART I

1. (5 points) Complete the following **definitions**.

(a) A *subspace* of a vector space V is a subset H of V that has the following properties:

(b) Let V be a vector space. A set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a *basis* for V if

2. (7 points) Let

$$A = \begin{bmatrix} 1 & -1 & 3 & 2 & 3 \\ -11 & 11 & -36 & -7 & 0 \\ 21 & -21 & 69 & 13 & -1 \\ -15 & 15 & -48 & -7 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis for Col A.

(b) What is rank A?

(c) What is $\dim \operatorname{Nul} A$?

(d) Find a basis for Nul A.

3. (4 points) Recall that $M_{2\times 2}$ is the set of 2×2 matrices. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Given that $\mathcal{B}$ is a basis for $M_{2\times 2}$, for $a, b, c, d \in \mathbb{R}$ and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2},$$

find $[A]_{\mathcal{B}}$.

4. (4 points) Define a linear transformation $T : M_{2\times 2} \to \mathbb{R}$ by

$$T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11} + a_{22}.$$

(a) Find matrices A_1, A_2, and A_3 in $M_{2\times 2}$ that span the kernel of T.

(b) Describe the range of T.

5. (5 points) For each of the following, fill in the blank using the appropriate letter from the list below so that the given statement is *always* true. You do **not** need to justify your answer.

(A) a subspace of $\mathbb{R}^m$
(B) a subspace of $\mathbb{R}^n$
(C) at most n vectors
(D) at least n vectors
(E) none of the above makes the statement always true

(a) If A is an $m \times n$ matrix, then Col A is ____________.

(b) If A is an $m \times n$ matrix, then Nul A is ____________.

(c) If V is a vector space with basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any linearly independent set in V has ____________.

(d) If V is a vector space with basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set of vectors in V that spans V has ____________.

(e) If V is a vector space, then any basis for V has ____________.

6. (10 points) For each of the following statements, circle either **T** for "True" or **F** for "False." You do **not** need to justify your answers. **Your score on this problem is the bigger of** 0 **and the difference between the number you answer correctly and the number you answer incorrectly**.

T F (a) If V is a finite-dimensional vector space and $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a set of vectors in V such that $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = V$, then some subset of S is a basis for V.

T F (b) A plane in $\mathbb{R}^3$ is a two-dimensional subspace of $\mathbb{R}^3$.

T F (c) If A is an $m \times n$ matrix and rank $A = m$, then the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ is onto.

T F (d) Some change-of-coordinates matrices are not invertible.

T F (e) A vector space is infinite-dimensional if it is spanned by an infinite set.

T F (f) A plane in $\mathbb{R}^3$ can be isomorphic to $\mathbb{R}^2$.

T F (g) If V is an n-dimensional ($n \geq 1$ an integer) vector space, then $V \cong \mathbb{R}^n$.

T F (h) The set

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for the set H of vectors in $\mathbb{R}^3$ whose first and second entries are equal.

T F (i) If A and B are $n \times n$ matrices, then $\det(A + B) = \det A + \det B$.

T F (j) If A is an invertible $n \times n$ matrix, then $(\det A)(\det A^{-1}) = 1$.

PART II

Your solutions to this part of the exam are due at the beginning of class tomorrow (November 3). Fully justify your solutions, and write them *neatly* on $8.5'' \times 11''$ sheets of paper; if you use more than one sheet of paper, staple your pages together in the upper left-hand corner. Use only a (non-colored) pencil or a pen with either blue or black ink.

You may refer only to your text and class notes. Otherwise, you are to work alone and to neither receive nor provide assistance to anyone else.

7. (9 points) Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} \right\}$$

and

$$\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

(a) Show that $\mathcal{B}$ and $\mathcal{C}$ are bases for $\mathbb{R}^3$.

(b) Let

$$\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}.$$

Find $[\mathbf{v}]_{\mathcal{B}}$.

(c) Find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

8. (7 points) Let $W = \{A \in M_{3\times 3} : A^T = -A\}$.

(a) Show that W is a subspace of $M_{3\times 3}$.

(b) Find a basis for W, and compute the dimension of W.

9. (7 points) Let

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

Assume that $\mathcal{B}$ is a basis for $\mathbb{R}^3$. Suppose $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also a basis for $\mathbb{R}^3$ and that the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$ is

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 3 & -1 & 1 \\ 2 & -2 & -1 \\ -2 & 1 & 3 \end{bmatrix}.$$

Find the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

10. (5 points) Let $T : \mathbb{P}_3 \to \mathbb{R}^2$ be given by

$$T(p(t)) = \begin{bmatrix} p(0) \\ p(-1) \end{bmatrix}.$$

(a) Show that T is a linear transformation.

(b) What is the kernel of T?

11. (7 points) Suppose V is a finite-dimensional, non-zero vector space with zero vector $\mathbf{0}$. Let U be a 3-dimensional subspace of V and W a 2-dimensional subspace of V such that $U \cap W = \{\mathbf{0}\}$.

(a) Show that if $\mathcal{B}$ is a basis for U and $\mathcal{C}$ is a basis for W, then $\mathcal{B} \cup \mathcal{C}$ is a basis for $U \cup W$.

(b) Show that every vector $\mathbf{x} \in U \cup W$ can be written as $\mathbf{x} = \mathbf{u} + \mathbf{w}$, where $\mathbf{u} \in U$ and $\mathbf{w} \in W$.

Name: ____

Math 22

2006-11-02
Exam II

Instructions: This exam has two parts. The first part you must complete during class today (November 2). Your solutions to the second part are due at the beginning of class tomorrow (November 3). Your solutions to the second part should be fully justified and *neatly* written on $8.5'' \times 11''$ sheets of paper; if you use more than one sheet of paper, staple your pages together in the upper left-hand corner. Use only a (non-colored) pencil or a pen with either blue or black ink.

You may refer to your text and class notes only on the take-home part of this exam. Otherwise, you are to work alone and to neither receive nor provide assistance to anyone else.

FERPA release: You must sign in the spot indicated below in order for me to be able to return your exam in lecture. Otherwise, you will have to pick up your exam in my office after the exams have been returned in lecture.

Sign here: ____.

Problem	Points	Score
1	5	
2	7	
3	4	
4	4	
5	5	
6	10	
7	9	
8	7	
9	7	
10	5	
11	7	
Total	70	