Application of Linear Algebra to Economics

- Wassily Leontief
  - divided U.S. economy into 500 sectors (e.g. coal industry, automotive industry, communications)
  - for each sector, wrote linear equation describing how sector distributes output to other sectors
- Leontief "input-output" (or "production") model

Terminology

- n: number of sectors in nation's economy
- $\mathbf{x} \in \mathbb{R}^n$ *production vector*: output of each sector for year
- $\mathbf{d} \in \mathbb{R}^n$ *final demand vector*: value of goods and services demanded from sectors by non-productive part of economy
- *intermediate demand*: inputs producers need for production

Leontief's question: is there a production level such that the total amount produced equals the total demand for production?

Is there an $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x} = \text{intermediate demand} + \mathbf{d}$?

The Model

- hold prices of goods and services constant
- measure unit of input and output in millions of dollars
- basic assumption: for each sector, there is a *unit consumption vector* **c** listing inputs needed per unit of output of sector

Example ($n = 3$)

	Inputs Consumed per Unit of Output		
Purchased from	Manufacturing	Agriculture	Services
Manufacturing	0.50	0.40	0.20
Agriculture	0.20	0.30	0.10
Services	0.10	0.10	0.30
	↑	↑	↑
	$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$

What will the manufacturing sector consume if it produces 100 units?

50 units from manufacturing, 20 units from agriculture, 10 units from services

Suppose sector

- has unit consumption vector $\mathbf{c}$
- produces x units of output

What is sector's intermediate demand? $x\mathbf{c}$

total intermediate demand $= x_1\mathbf{c}_1 + \cdots + x_n\mathbf{c}_n = C\mathbf{x}$, where C is *consumption matrix* $C = [\mathbf{c}_1 \cdots \mathbf{c}_n]$

Leontief's question: is there an $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x} = C\mathbf{x} + \mathbf{d}$? Alternatively, is there an $\mathbf{x} \in \mathbb{R}^n$ such that $(I_n - C)\mathbf{x} = \mathbf{d}$?

Example ($n = 3$)

	Inputs Consumed per Unit of Output		
Purchased from	Manufacturing	Agriculture	Services
Manufacturing	0.50	0.40	0.20
Agriculture	0.20	0.30	0.10
Services	0.10	0.10	0.30
	↑	↑	↑
	$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$

$$C = \begin{bmatrix} 0.50 & 0.40 & 0.20 \\ 0.20 & 0.30 & 0.10 \\ 0.10 & 0.10 & 0.30 \end{bmatrix}$$

Example $(n = 3)$

Suppose the final demand is 50 units for manufacturing, 30 units for agriculture, and 20 units for services. What is the production level that will satisfy this demand?

$$I_3 - C = \begin{bmatrix} 0.50 & -0.40 & -0.20 \\ -0.20 & 0.70 & -0.10 \\ -0.10 & -0.10 & 0.70 \end{bmatrix}$$

$$\begin{bmatrix} 0.50 & -0.40 & -0.20 & 50 \\ -0.20 & 0.70 & -0.10 & 30 \\ -0.10 & -0.10 & 0.70 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 225.9 \\ 0 & 1 & 0 & 118.5 \\ 0 & 0 & 1 & 77.8 \end{bmatrix}$$

- $I_n - C$ invertible implies $\mathbf{x} = (I_n - C)^{-1}\mathbf{d}$
- in most practical cases, $I_n - C$ is invertible

column sum: sum of entries in column

<u>Theorem</u>: Let C be the consumption matrix for an economy and $\mathbf{d}$ the final demand vector. If C and $\mathbf{d}$ have non-negative entries and if each column sum of C is less than 1, then $I - C$ is invertible, and the production vector

$$\mathbf{x} = (I - C)^{-1}\mathbf{d}$$

has non-negative entries and is the unique solution of

$$\mathbf{x} = C\mathbf{x} + \mathbf{d}.$$

Note: sector should need less than one unit's worth of inputs to produce one unit of output, so column sums of consumption matrix should all be less than 1

- suppose $\mathbf{d}$ is presented to various sectors at start of year and sectors set $\mathbf{x} = \mathbf{d}$
- intermediate demand $= C\mathbf{d}$
- to meet demand of $C\mathbf{d}$, sectors need inputs of $C(C\mathbf{d}) = C^2\mathbf{d}$, creating second round of intermediate demand of $C(C^2\mathbf{d}) = C^3\mathbf{d}$
- theoretically, process continues indefinitely

	Demand	Inputs Needed
Final Demand	$\mathbf{d}$	$C\mathbf{d}$
Intermediate demand		
round 1	$C\mathbf{d}$	$C(C\mathbf{d}) = C^2\mathbf{d}$
round 2	$C^2\mathbf{d}$	$C(C^2\mathbf{d}) = C^3\mathbf{d}$
round 3	$C^3\mathbf{d}$	$C(C^3\mathbf{d}) = C^4\mathbf{d}$
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} \mathbf{x} &= \mathbf{d} + C\mathbf{d} + C^2\mathbf{d} + C^3\mathbf{d} + \cdots \\ &= (I_n + C + C^2 + C^3 + \cdots)\mathbf{d} \end{aligned}$$

- $(I_n - C)(I_n + C + C^2 + \cdots + C^m) = I_n - C^{m+1}$
- if all column sums of C are less than 1, then
  - $I_n - C$ is invertible
  - $C^m \to 0$ as $m \to \infty$
  - $I_n - C^{m+1} \to I_n$ as $m \to \infty$ (idea: $0 < t < 1$ implies $t^m \to 0$ as $m \to \infty$)
- $(I_n - C)^{-1} \approx I_n + C + C^2 + \cdots + C^m$; i.e., right-hand side can be made as close to $(I_n - C)^{-1}$ as we want by taking m large enough

- in actual input-output models, powers of consumption matrix C approach 0 quickly, and for given final demand $\mathbf{d}$, vectors $C^m\mathbf{d}$ approach $\mathbf{0}$ quickly
- entries in $(I_n - C)^{-1}$ can be used to predict how production $\mathbf{x}$ will have to change when $\mathbf{d}$ changes: entries in column j of $(I_n - C)^{-1}$ are increased amounts various sectors will have to produce to satisfy increase of one unit in final demand for output from sector j