The Invertible Matrix Theorem

Let A be an $n \times n$ matrix. Then the following are equivalent:

a. The matrix A is invertible (non-singular).

b. The matrix A is row equivalent to I_n.

c. The matrix A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. For each $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

j. There is an $n \times n$ matrix C such that $CA = I_n$.

k. There is an $n \times n$ matrix D such that $AD = I_n$.

l. The matrix A^T is invertible.

m. The columns of A form a basis for $\mathbb{R}^n$.

n. The column space of A is $\mathbb{R}^n$ ($\text{Col } A = \mathbb{R}^n$).

o. The dimension of the column space of A is n ($\dim \text{Col } A = n$).

p. The rank of A is n ($\text{rank } A = n$).

q. The null space of A is $\{\mathbf{0}\}$ ($\text{Nul } A = \{\mathbf{0}\}$).

r. The dimension of the null space of A is 0 ($\dim \text{Nul } A = 0$).

s. The number 0 is not an eigenvalue of A.

t. The determinant of A is not zero ($\det A \neq 0$).

u. The orthogonal complement of the column space of A is $\{\mathbf{0}\}$ ($(\text{Col } A)^\perp = \{\mathbf{0}\}$).

v. The orthogonal complement of the null space of A is $\mathbb{R}^n$ ($(\text{Nul } A)^\perp = \mathbb{R}^n$).

w. The row space of A is $\mathbb{R}^n$ ($\text{Row } A = \mathbb{R}^n$).

x. The matrix A had n non-zero singular values.