MATH 22 LINEAR ALGEBRA FALL '04
HOMEWORK #9 ANSWER KEY

5.4: 4, 6, 8, 10, 12, 16

(4.) $M = [\,T(b_1) \quad T(b_2) \quad T(b_3)\,]$

$$= \begin{bmatrix} 2 & -4 & 5 \\ 0 & -1 & 3 \end{bmatrix}$$

(6.) $T : \mathbb{P}_2 \to \mathbb{P}_4 : p(t) \mapsto p(t) + t^2 p(t)$

(a.) $T(2 - t + t^2) = (2 - t + t^2) + t^2(2 - t + t^2)$

$= 2 - t + t^2 + 2t^2 - t^3 + t^4 = 2 - t + 3t^2 - t^3 + t^4$

(b.) LET $p(t), q(t) \in \mathbb{P}_2$ AND $c \in \mathbb{R}$.

$T(p(t) + q(t)) = (p(t) + q(t)) + t^2(p(t) + q(t))$

$= (p(t) + t^2 p(t)) + (q(t) + t^2 q(t))$
$= T(p(t)) + T(q(t))$

$T(c\,p(t)) = c\,p(t) + t^2 c\,p(t) = c(p(t) + t^2 p(t))$
$= c\,T(p(t))$ AND THUS T IS LINEAR.

(c.) $M = [\,[T(1)]_\beta \; [T(t)]_\beta \; [T(t^2)]_\beta\,]$ WHERE $\beta = \{1, t, t^2, t^3, t^4\}$.

$T(1) = [1 + t^2]_\beta$, $T(t) = [t + t^3]_\beta$, $T(t^2) = [t^2 + t^4]_\beta$

THUS $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(8.) LET $x = 3b_1 - 4b_2$

THEN $[x]_\beta = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$

AND $[T(x)]_\beta = [T]_\beta [x]_\beta$

$$= \begin{bmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 24 \\ -20 \\ 11 \end{bmatrix}$$

THUS $T(x) = 24b_1 - 20b_2 + 11b_3$.

(10.) (a.) LET $p(t), q(t) \in \mathbb{P}_3$ AND $c \in \mathbb{R}$.

$$T(p(t) + q(t)) = T((p+q)(t))$$

$$= \begin{bmatrix} (p+q)(-3) \\ (p+q)(-1) \\ (p+q)(1) \\ (p+q)(3) \end{bmatrix} = \begin{bmatrix} p(-3) + q(-3) \\ p(-1) + q(-1) \\ p(1) + q(1) \\ p(3) + q(3) \end{bmatrix}$$

$$= \begin{bmatrix} p(-3) \\ p(-1) \\ p(1) \\ p(3) \end{bmatrix} + \begin{bmatrix} q(-3) \\ q(-1) \\ q(1) \\ q(3) \end{bmatrix} = T(p(t)) + T(q(t)).$$

$$T(cp(t)) = T((cp)(t)) = \begin{bmatrix} (cp)(-3) \\ (cp)(-1) \\ (cp)(1) \\ (cp)(3) \end{bmatrix} = \begin{bmatrix} c\,p(-3) \\ c\,p(-1) \\ c\,p(1) \\ c\,p(3) \end{bmatrix}$$

$$= c \begin{bmatrix} p(-3) \\ p(-1) \\ p(1) \\ p(3) \end{bmatrix} = c\,T(p(t))$$ AND THUS T IS LINEAR.

(b.) $M = \begin{bmatrix} T(1) & T(t) & T(t^2) & T(t^3) \end{bmatrix}$

$T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $T(t) = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix}$

$T(t^2) = \begin{bmatrix} 9 \\ 1 \\ 1 \\ 9 \end{bmatrix}$ $T(t^3) = \begin{bmatrix} -27 \\ -1 \\ 1 \\ 27 \end{bmatrix}$

THUS $M = \begin{bmatrix} 1 & -3 & 9 & -27 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \end{bmatrix}$.

(12.) $M = \begin{bmatrix} [T(b_1)]_\beta & [T(b_2)]_\beta \end{bmatrix}$

$T(b_1) = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$T(b_2) = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

$\begin{bmatrix} 3 & -1 & 5 \\ 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 6 & -2 & 10 \\ -6 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 6 & -2 & 10 \\ 0 & -5 & 10 \end{bmatrix}$

$\sim \begin{bmatrix} 3 & -1 & 5 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 3 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

THUS $[T(b_1)]_\beta = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\begin{bmatrix} 3 & -1 & 5 \\ 2 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 6 & -2 & 10 \\ -6 & -3 & -15 \end{bmatrix} \sim \begin{bmatrix} 6 & -2 & 10 \\ 0 & -5 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -1 & 5 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 6 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

THUS $[T(b_2)]_\beta = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

SO $M = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

(16.) WE DIAGONALIZE A AND APPLY THEOREM 8.

$\det(A - \lambda I) = (2-\lambda)(3-\lambda) - 6$

$= (\lambda - 2)(\lambda - 3) - 6 = \lambda^2 - 5\lambda$

THUS THE EIGENVALUES OF A ARE THE ROOTS OF $\lambda^2 - 5\lambda$.

$\lambda^2 - 5\lambda = 0 \Rightarrow \lambda(\lambda - 5) = 0$

$\Rightarrow \lambda = 5, 0$.

$E_5(A) = \text{NUL}(A - 5I) = \text{NUL} \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix}$

$= \text{SPAN}\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$

$E_0(A) = \text{NUL}(A - 0I) = \text{NUL}(A) = \text{NUL} \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix}$

$= \text{SPAN}\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

THUS $\begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}^{-1}$

LET $\beta = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$ THEN $[T]_\beta = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$ DIAGONAL.

6.1 : 2, 16, 18, 30

(2.) $w \cdot w = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = 3^2 + (-1)^2 + (-5)^2 = 35$

$x \cdot w = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = 6(3) + (-2)(-1) + 3(-5) = 5$

$\frac{x \cdot w}{w \cdot w} = \frac{5}{35} = \frac{1}{7}$

(16.) $u \cdot v = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = 12(2) + 3(-3) + (-5)(3) = 0$

THUS u AND v ARE ORTHOGONAL.

(18.) $y \cdot z = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix} = (-3)(1) + (7)(-8) + 4(15) + 0(-7)$

$= -3 - 56 + 60 = 1$

THUS y AND z ARE NOT ORTHOGONAL.

(30.) (a.) $(cz) \cdot u = c(z \cdot u) = c \cdot 0 = 0$

THUS cz IS ORTHOGONAL TO u.

(b.) $(z_1 + z_2) \cdot u = z_1 \cdot u + z_2 \cdot u = 0 + 0 = 0.$

THUS $z_1 + z_2$ IS ORTHOGONAL TO u.

(c.) CLEARLY, $0 \in W^T$ SINCE $0 \cdot u = 0 \quad \forall u$

WE HAVE NOW SHOWN THAT THE THREE AXIOMS HOLD

SO $W^\perp$ IS A SUBSPACE BY DEFINITION P. 220.

6.7 : 4, 6, 14, 18

(4.) $\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$

$\langle 3t - t^2, 3 + 2t^2 \rangle = (-4)(5) + 0(3) + 2(5) = -10$

(6.) $\|p\| = \sqrt{\langle p, p \rangle} = \sqrt{p(-1)^2 + p(0)^2 + p(1)^2}$

$= \sqrt{(-4)^2 + 0^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$

$\|q\| = \sqrt{\langle q, q \rangle} = \sqrt{q(-1)^2 + q(0)^2 + q(1)^2} = \sqrt{5^2 + 3^2 + 5^2}$

$= \sqrt{59}$

(14.) $T : V \to \mathbb{R}^n$ 1-1 LINEAR TRANSFORMATION

$\langle u, v \rangle = T(u) \cdot T(v)$ WE SHOW THE AXIOMS ARE SATIFIED:

(i.) $\langle u, v \rangle = T(u) \cdot T(v) = T(v) \cdot T(u) = \langle v, u \rangle$

(ii.) $\langle u + v, w \rangle = T(u + v) \cdot T(w) = (T(u) + T(v)) \cdot T(w)$
$= T(u)\,T(w) + T(v)\,T(w) = \langle u, w \rangle + \langle v, w \rangle$

(iii.) $\langle cu, v \rangle = T(cu) \cdot T(v) = c\,T(u) \cdot T(v) = c\langle u, v \rangle$

(iv.) $\langle u, u \rangle = T(u) \cdot T(u) = \|T(u)\|^2 \geq 0$

AND $\|T(u)\|^2 = 0$ IFF $\|T(u)\| = 0$
IFF $T(u) = 0$ IFF $u = 0$
SINCE T IS 1-1.

QED

(18) $\|u+v\|^2 + \|u-v\|^2 = (u+v)\cdot(u+v) + (u-v)\cdot(u-v)$

$= u\cdot u + u\cdot v + v\cdot u + v\cdot v$
$+ u\cdot u - u\cdot v - v\cdot u + v\cdot v$
$= u\cdot u + v\cdot v + u\cdot u + v\cdot v$
$= 2u\cdot u + 2v\cdot v = 2\|u\|^2 + 2\|v\|^2$

QED

6.2: 6, 10, 14, 16, 20, 34

(6.) NOT ORTHOGONAL BECAUSE

$$\begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix} = -12+3-15-8 = -32 \neq 0.$$

(10.) $\{u_1, u_2, u_3\}$ IS AN ORTHOGONAL SET BECAUSE $u_1 \cdot u_2 = u_2 \cdot u_3 = u_1 \cdot u_3 = 0$.
IT FOLLOWS THAT u_1, u_2, u_3 ARE LINEARLY INDEPENDENT AND THEREFORE FORM A BASIS FOR $\mathbb{R}^3$. THUS $\{u_1, u_2, u_3\}$ IS AN ORTHOGONAL BASIS FOR $\mathbb{R}^3$.

WE HAVE TWO WAYS TO EXPRESS X AS A LINEAR COMBINATION OF u_1, u_2, u_3:

$$\begin{bmatrix} 3 & 2 & 1 & 5 \\ -3 & 2 & 1 & -3 \\ 0 & -1 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 1 & 5 \\ 0 & 4 & 2 & 2 \\ 0 & -1 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 1 & 5 \\ 0 & 4 & 2 & 2 \\ 0 & -4 & 16 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 2 & 1 & 5 \\ 0 & 4 & 2 & 2 \\ 0 & 0 & 18 & 6 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 1 & 5 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 0 & \frac{14}{3} \\ 0 & 2 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 2 & 0 & \frac{14}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 0 & 4 \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

$$\Rightarrow \quad x = \frac{4}{3} u_1 + \frac{1}{3} u_2 + \frac{1}{3} u_3.$$

ALTERNATIVELY, $x = c_1 u_1 + c_2 u_2 + c_3 u_3$

WHERE $c_1 = \dfrac{x \cdot u_1}{u_1 \cdot u_1} = \dfrac{4}{3}$, $c_2 = \dfrac{x \cdot u_2}{u_2 \cdot u_2} = \dfrac{1}{3}$, $c_3 = \dfrac{x \cdot u_3}{u_3 \cdot u_3} = \dfrac{1}{3}$

(BY THEOREM 5).

(14.) $y = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ $\quad u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ $\quad L = \text{SPAN}\{u\}$

$$\hat{y} = \text{proj}_L\, y = \frac{y \cdot u}{u \cdot u} u = \frac{20}{50} u = \frac{2}{5} u = \frac{2}{5}\begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

$$z = y - \hat{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$$

SO $y = \hat{y} + z = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$.

(16.) WE HAVE TWO WAYS TO SOLVE THIS PROBLEM. GIVEN A POINT (x_1, y_1) AND A LINE $ax + by + c = 0$, THE DISTANCE FROM THE POINT TO THE LINE IS GIVEN BY $d = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$.

IN THIS CASE, $(x_1, y_1) = (-3, 9)$ AND THE LINE HAS EQUATION $2x - y = 0$,

SO $$d = \frac{|2(-3) + (-1)(9)|}{\sqrt{5}} = \frac{15}{\sqrt{5}} = 3\sqrt{5}.$$

ALTERNATIVELY, WE WRITE $y = \hat{y} + z$ WHERE $\hat{y} \in \text{SPAN}\{u\}$ AND $z \in \text{SPAN}\{u\}^{\perp}$.

THEN $d = \|z\|$.

$$\hat{y} = \frac{y \cdot u}{u \cdot u} u = \frac{15}{5} u = 3u = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \in \text{span}\{u\}$$

$$z = y - \hat{y} = \begin{bmatrix} -3 \\ 9 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix} \in \text{span}\{u\}^{\perp}$$

$$d = \|z\| = \sqrt{45} = 3\sqrt{5}.$$

(20.) LET $u_1 = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$ $u_2 = \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}$

Then $\{u_1, u_2\}$ is orthogonal because $u_1 \cdot u_2 = 0$ but not orthonormal because

$\|u_1\| = 1$ BUT $\|u_2\| = \frac{\sqrt{5}}{3} \neq 1$

However, $\{u_1, \frac{3}{\sqrt{5}} u_2\} =$

$$\left\{ \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \right\}$$ is orthonormal.

(34.) $\text{refl}_L y = 2 \cdot \text{proj}_L y - y = \frac{2\, y \cdot u}{u \cdot u} u - y$

LET $x, y \in \mathbb{R}^n$, $c \in \mathbb{R}$.

$$\text{refl}_L (x+y) = \frac{2(x+y) \cdot u}{u \cdot u} u - (x+y)$$

$$= \left(\frac{2x \cdot u}{u \cdot u} u - x\right) + \left(\frac{2y \cdot u}{u \cdot u} - y\right) = \text{refl}_L x + \text{refl}_L y$$

$$\text{refl}_L (cx) = \frac{2(cx) \cdot u}{u \cdot u} - cx = c\left(\frac{2x \cdot u}{u \cdot u} - x\right) = c\, \text{refl}_L x$$

✓

6.3 : 2, 6, 10, 12, 16, 18, 24

(2.) BY THEOREM 8, WE HAVE $v = \hat{v} + z$
WHERE $\hat{v} \in \text{SPAN}\{u_1\}$ AND $z \in \text{SPAN}\{u_2, u_3, u_4\}$.

$$\hat{v} = \frac{v \cdot u_1}{u_1 \cdot u_1} u_1 = \frac{14}{7} u_1 = 2u_1 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix}$$

AND $z = v - \hat{v} = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -5 \\ 1 \end{bmatrix}$,

(6.) $u_1 \cdot u_2 = -4(0) + -1(1) + 1(1) = 0$
THUS $\{u_1, u_2\}$ IS AN ORTHOGONAL SET.

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{-27}{18} u_1 + \frac{5}{2} u_2$$

$$= -\frac{3}{2} \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}.$$

(10.) BY THEOREM 8, $y = \hat{y} + z$ WHERE $\hat{y} \in W$ AND $z \in W^{\perp}$.

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{y \cdot u_3}{u_3 \cdot u_3} u_3$$

$$= \frac{1}{3} u_1 + \frac{14}{3} u_2 - \frac{5}{3} u_3$$

$$= \frac{1}{3}(u_1 + 14u_2 - 5u_3)$$

$$= \frac{1}{3}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 14 \\ 0 \\ 14 \\ 14 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \\ -5 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}.$$

(12.) BY THEOREM 9, THE CLOSEST POINT TO y IN W

IS $\hat{y} = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2$

$$= \frac{30}{10} v_1 + \frac{26}{26} v_2 = 3v_1 + v_2$$

$$= \begin{bmatrix} 3 \\ -6 \\ -3 \\ 6 \end{bmatrix} + \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}.$$

(16.) $d = \|y - \hat{y}\|$

$$y - \hat{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix} - \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$$

THUS $d = \sqrt{4^2 + 4^2 + 4^2 + 4^2} = 8$.

(18.) (a.) $U^T U = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} \end{bmatrix} = \frac{1}{10} + \frac{9}{10} = 1.$

$$UU^T = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{9}{10} \end{bmatrix}.$$

(b.) $\text{proj}_W y = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 = \frac{-\frac{20}{\sqrt{10}}}{1} u_1 = -\frac{20}{\sqrt{10}} u_1$

$= -2\sqrt{10}\, u_1 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$

$$(UU^T)y = \frac{1}{10} \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} 7 \\ 9 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} -20 \\ 60 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

(24.) (a.) TO SHOW $\{w_1, \ldots, w_p, v_1, \ldots, v_q\}$ IS AN ORTHOGONAL SET, WE SHOW THAT ITS ELEMENTS ARE PAIRWISE ORTHOGONAL. THIS IS CLEARLY THE CASE, BECAUSE IF WE PICK ANY TWO ELEMENTS, THEY ARE EITHER BOTH w'S, BOTH v'S, OR ONE IS A w AND THE OTHER IS A v. IN EITHER CASE, THEY ARE ORTHOGONAL.

(b.) THIS FOLLOWS DIRECTLY FROM THEOREM 8 (THE ORTHOGONAL DECOMPOSITION THEOREM).

(c.) BY PARTS a AND b, $\{w_1, \ldots, w_p, v_1, \ldots, v_q\}$ IS A BASIS FOR $\mathbb{R}^n$, AND THEREFORE $p + q = n$, i.e. $\dim W + \dim W^{\perp} = n$.