MATH 22 LINEAR ALGEBRA FALL '04
HOMEWORK #8 ANSWER KEY

5.2 : 4, 10, 12, 16, 20, 24

(4.) CHARACTERISTIC POLYNOMIAL:

$$CH_A(\lambda) = \det(A - \lambda I) = (5-\lambda)(3-\lambda) - 12$$
$$= \lambda^2 - 8\lambda + 3$$

EIGENVALUES: $\lambda = \dfrac{8 \pm \sqrt{64 - 4(1)(3)}}{2(1)}$

$$= \frac{8 \pm \sqrt{52}}{2} = 4 \pm \sqrt{13}$$

SO THE TWO EIGENVALUES ARE $4 + \sqrt{13}$ AND $4 - \sqrt{13}$.

(10.) $CH_A(\lambda) = \det \begin{bmatrix} -\lambda & 3 & 1 \\ 3 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{bmatrix}$

$$= -\lambda(\lambda^2 - 4) - 3(-3\lambda - 2) + 1(6 + \lambda)$$
$$= -\lambda^3 + 4\lambda + 9\lambda + 6 + 6 + \lambda$$
$$= -\lambda^3 + 14\lambda + 12.$$

(12.) $CH_A(\lambda) = \det \begin{bmatrix} -\lambda - 1 & 0 & 1 \\ -3 & -\lambda + 4 & 1 \\ 0 & 0 & -\lambda + 2 \end{bmatrix}$

$$= (-\lambda + 2)(\lambda + 1)(\lambda - 4) = -(\lambda - 2)(\lambda + 1)(\lambda - 4)$$
$$= -(\lambda^2 - \lambda - 2)(\lambda - 4)$$
$$= -(\lambda^3 - 5\lambda^2 + 2\lambda + 8)$$
$$= -\lambda^3 + 5\lambda^2 - 2\lambda - 8$$

(16.) 5, −4, 1, 1

(20.) $CH_A(\lambda) = \det(A - \lambda I) = \det(A - \lambda I)^T$

$$= \det(A^T - \lambda I^T) = \det(A^T - \lambda I) = CH_{A^T}(\lambda).$$

(24.) A, B SIMILAR $\Rightarrow A = PBP^{-1} \Rightarrow$

$\det A = \det(PBP^{-1}) = (\det P)(\det B)(\det P^{-1})$
$= (\det P)(\det P^{-1})(\det B) = (\det PP^{-1})(\det B)$
$= (\det I)(\det B) = 1 \cdot \det B = \det B.$

5.3: 2, 4, 6, 8, 16, 24, 32

(2.) $A^4 = (PDP^{-1})^4 = PD^4P^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/16 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$

$= \frac{1}{16}\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 16 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 32 & -3 \\ -48 & 5 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$

$= \frac{1}{16}\begin{bmatrix} 151 & 90 \\ -225 & -134 \end{bmatrix}.$

(4.) $A^k = (PDP^{-1})^k = PD^kP^{-1} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2^k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$

$= \begin{bmatrix} 3\cdot 2^k & 4 \\ 2^k & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 4-3(2^k) & 12(2^k-1) \\ 1-2^k & 4(2^k)-3 \end{bmatrix}.$

(6.) THE EIGENVALUES ARE 4, 5

$E_4(A) = \text{SPAN}\left\{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\right\}$

$E_5(A) = \text{SPAN}\left\{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

(8.) REPEATED EIGENVALUE 5.
$E_5(A) = \text{NUL}(A-5I) = \text{NUL}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \text{SPAN}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}.$

A DOES NOT HAVE TWO LINEARLY INDEPENDENT EIGENVECTORS, THUS A IS NOT DIAGONALIZABLE, BY THE DIAGONALIZATION THEOREM.

(16.) $A = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$

EIGENVALUES ARE 1, 2.

$E_1(A) = \text{NUL}(A - I) = \text{NUL} \begin{bmatrix} -1 & -4 & -6 \\ -1 & -1 & -3 \\ 1 & 2 & 4 \end{bmatrix}$

$$\begin{bmatrix} -1 & -4 & -6 & 0 \\ -1 & -1 & -3 & 0 \\ 1 & 2 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 6 & 0 \\ -1 & -1 & -3 & 0 \\ -1 & -2 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 6 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 6 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -2x_3 \\ x_2 = -x_3 \\ x_3 \text{ FREE} \end{cases}$$

THUS $E_1(A) = \text{SPAN} \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

$E_2(A) = \text{NUL}(A - 2I) = \text{NUL} \begin{bmatrix} -2 & -4 & -6 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{bmatrix}$

$$\begin{bmatrix} -2 & -4 & -6 & 0 \\ -1 & -2 & -3 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -2x_2 - 3x_3 \\ x_2, x_3 \text{ FREE} \end{cases}$$

THUS $E_2(A) = \text{SPAN} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

BY THE DIAGONALIZATION THEOREM, A IS DIAGONALIZABLE:

$$\begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -2 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & -2 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}.$$

<u>CHECK</u>: $\begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} -2 & -2 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -2 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$

(24.) No, BY THEOREM 7b, P. 324.

(32.) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ IS A NONDIAGONAL 2×2 MATRIX THAT IS DIAGONALIZABLE BUT NOT INVERTIBLE.
A IS NOT INVERTIBLE BECAUSE $\det A = 0$.
TO SEE THAT A IS DIAGONALIZABLE, OBSERVE THAT A HAS EIGENVALUES 0, 2.

$$E_0(A) = \text{NUL } A = \text{SPAN}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

$$E_2(A) = \text{NUL}(A - 2I) = \text{SPAN}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

THUS A IS DIAGONALIZABLE BY THE DIAGONALIZATION THEOREM AND WE HAVE:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}.$$

<u>CHECK</u>: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$

<u>NOTE</u>: TO SHOW THAT A IS DIAGONALIZABLE, WE NEED ONLY TO OBSERVE THAT IT HAS 2 DISTINCT EIGENVALUES. THE REST OF THE COMPUTATIONS ARE NECESSARY TO ACTUALLY DIAGONALIZE A.

5.3 : 10, 14, 20, 26, 28

(10.) $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

<u>CHARACTERISTIC EQUATION</u>:

$$CH_A(\lambda) = (2-\lambda)(1-\lambda) - 12 = \lambda^2 - 3\lambda - 10$$
$$= (\lambda - 5)(\lambda + 2).$$

<u>EIGENVALUES</u>: 5, −2

$$E_5(A) = \text{NUL}(A - 5I) = \text{NUL}\begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix}$$
$$= \text{SPAN}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$$E_{-2}(A) = \text{NUL}(A + 2I) = \text{NUL}\begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix}$$
$$= \text{SPAN}\left\{\begin{bmatrix} 3 \\ -4 \end{bmatrix}\right\}.$$

THUS A IS DIAGONALIZABLE BY THE DIAGONALIZATION THEOREM:

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix}^{-1}.$$

<u>CHECK</u>: $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}.$

(14.) $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

<u>EIGENVALUES</u>: 4, 5

$E_4(A) = \text{NUL}(A-4I) = \text{NUL}\begin{bmatrix} 0 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 0 & -2 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 = -\frac{1}{2} x_2 \\ x_2 \text{ FREE} \\ x_3 = 0 \end{cases} \Rightarrow E_4(A) = \text{SPAN}\left\{\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right\}.$$

$E_5(A) = \text{NUL}(A-5I) = \text{NUL}\begin{bmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} -1 & 0 & -2 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -2x_3 \\ x_2, x_3 \text{ FREE} \end{cases}$$

$$\Rightarrow E_5(A) = \text{SPAN}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right\}$$

SO BY THE DIAGONALIZATION THEOREM, A IS DIAGONALIZABLE:

$$\begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}.$$

<u>CHECK</u>:

$$\begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

(26.) $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

EIGENVALUES: 4, 2 (EACH OF MULTIPLICITY 2).

$E_4(A) = \text{NUL}(A - 4I) = \text{NUL} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = 2x_4 \\ x_2, x_4 \text{ FREE} \\ x_3 = 0 \end{cases}$

$\Rightarrow E_4(A) = \text{SPAN} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$

$E_2(A) = \text{NUL}(A - 2I) = \text{NUL} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = x_2 = 0 \\ x_3, x_4 \text{ FREE} \end{cases}$

$\Rightarrow E_2(A) = \text{SPAN} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$

SO BY THE DIAGONALIZATION THEOREM, A IS DIAGONALIZABLE:

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}^{-1}$$

AND THIS MAY BE VERIFIED AS USUAL.

(26.) YES, IT IS POSSIBLE THAT A IS NOT DIAGONALIZABLE (BY THEOREM 7b P. 324) IF THE THIRD EIGENSPACE IS ONE-DIMENSIONAL.

(28.) LET A BE A $n \times n$ MATRIX.
A HAS n LINEARLY INDEPENDENT EIGENVECTORS IFF A^T HAS n LINEARLY INDEPENDENT EIGENVECTORS.
PROOF: THE BICONDITIONAL FOLLOWS AUTOMATICALLY FROM THE CONDITIONAL, SINCE $(A^T)^T = A$.
TO PROVE THE CONDITIONAL, IT SUFFICES TO PROVE THAT A IS DIAGONALIZABLE IMPLIES THAT A^T IS DIAGONALIZABLE, (SINCE THE DIAGONALIZATION THEOREM ASSERTS THAT A $n \times n$ MATRIX IS DIAGONALIZABLE IFF IT HAS n LINEARLY INDEPENDENT EIGENVECTORS.)

A DIAGONALIZABLE $\Rightarrow A = PDP^{-1}$

$$\Rightarrow A^T = (PDP^{-1})^T = (P^{-1})^T D^T P^T$$

$$= (P^T)^{-1} D ((P^T)^{-1})^{-1} \Rightarrow A^T \text{ DIAGONALIZABLE.}$$

QED