# MATH 22 LINEAR ALGEBRA FALL '04
## HOMEWORK #6 ANSWER KEY

4.2: 32, 34

(32.) $\mathbb{P}_2 = \{a_0 + a_1 t + a_2 t^2 : a_0, a_1, a_2 \in \mathbb{R}\}$

$T: \mathbb{P}_2 \to \mathbb{R}^2$ LINEAR, DEFINED BY

$$T(p) = \begin{bmatrix} p(0) \\ p(0) \end{bmatrix}$$

$$T(a_0 + a_1 t + a_2 t^2) = \begin{bmatrix} a_0 \\ a_0 \end{bmatrix} = a_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

THE KERNEL OF $\mathbb{P}_2$ IS THE SUBSPACE OF POLYNOMIALS IN $\mathbb{P}_2$ WITH CONSTANT TERM ZERO:
$\text{Ker}\, T = \{a_1 t + a_2 t^2 : a_1, a_2 \in \mathbb{R}\}$

THUS $\text{Ker}\, T = \text{Span}\{t, t^2\}$. IN FACT, $\{t, t^2\}$ IS A BASIS FOR $\text{Ker}\, T$.

THE RANGE OF $T$ IS $\text{SPAN}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$, THE LINE

$y = x$ IN $\mathbb{R}^2$.

(34.) LET $f \in C[0,1]$, WHICH MEANS THAT
$f : [0,1] \to \mathbb{R}$ IS A CONTINUOUS FUNCTION.
THEN $T(f) : [0,1] \to \mathbb{R}$ IS A CONTINUOUS FUNCTION
GIVEN BY

$$T(f)(x) = \int_0^x f(t)\,dt \quad \forall\, x \in [0,1]$$

SINCE THIS IS THE UNIQUE ANTIDERIVATIVE OF $f$
SATISFYING $T(f)(0) = 0$.
LET $f, g \in C[0,1]$.

$$T(f+g)(x) = \int_0^x (f+g)(t)\,dt = \int_0^x (f(t)+g(t))\,dt$$

$$= \int_0^x f(t)\,dt + \int_0^x g(t)\,dt = T(f)(x) + T(g)(x)$$

$$= (T(f)+T(g))(x) \quad \forall\, x \in [0,1]$$

THUS $T(f+g) = T(f) + T(g)$.

LET $c \in \mathbb{R}$. $T(cf)(x) = \int_0^x (cf)(t)\,dt$

$$= \int_0^x cf(t)\,dt = c\int_0^x f(t)\,dt = c\,T(f)(x) \quad \forall\, x \in [0,1]$$

THUS $T(cf) = c\,T(f)$.

THEREFORE $T : C[0,1] \to C[0,1]$ IS LINEAR.

THE KERNEL OF $T$ IS THE ZERO SUBSPACE
OF $C[0,1]$, i.e. THE SUBSPACE OF $C[0,1]$
CONSISTING OF ONLY ONE ELEMENT, THE
ZERO FUNCTION, DEFINED BY $f(x) = 0 \ \forall\, x \in [0,1]$.

4.3: 4, 8, 16, 20, 24, 26

(4.) $\begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -7 \\ 0 & -2 & -2 \\ 0 & \frac{3}{2} & \frac{15}{2} \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -7 \\ 0 & -6 & -6 \\ 0 & 6 & 30 \end{bmatrix}$

$\sim \begin{bmatrix} 2 & 1 & -7 \\ 0 & -6 & -6 \\ 0 & 0 & 24 \end{bmatrix}$ SO THIS SET IS A BASIS FOR $\mathbb{R}^3$ BY THE THEOREM STATED IN EXAMPLE 3.

(8.) THIS SET IS NOT A BASIS FOR $\mathbb{R}^3$ BECAUSE IT IS LINEARLY DEPENDENT.

$\begin{bmatrix} 1 & 0 & 3 & 0 \\ -4 & 3 & -5 & 2 \\ 3 & -1 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 3 & 7 & 2 \\ 0 & -1 & -5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 3 & 7 & 2 \\ 0 & -3 & -15 & -6 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 3 & 7 & 2 \\ 0 & 0 & -8 & -4 \end{bmatrix}$ SINCE THERE IS A PIVOT POSITION IN EACH ROW, THIS SET SPANS $\mathbb{R}^3$.

(16.) APPLYING THEOREM 6, WE HAVE:

$\begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 1 & 1 & -1 & -4 & 1 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 3 & -7 & -9 & 1 \\ 0 & -3 & 3 & 9 & -9 \\ 0 & -3 & 6 & 9 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 3 & -7 & -9 & 1 \\ 0 & 0 & -4 & 0 & -8 \\ 0 & 0 & -1 & 0 & -2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 3 & -7 & -9 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ THE PIVOT COLUMNS ARE THE FIRST THREE COLUMNS, SO HEEDING THE WARNING ON P. 242, A BASIS IS

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix} \right\}.$

(20.) BY THEOREM 5, ANY TWO VECTORS FROM THE SET $\{v_1, v_2, v_3\}$ FORM A BASIS FOR H, SINCE NO VECTOR IN THE SET IS A MULTIPLE OF ANY OTHER VECTOR IN THE SET.

(24.) LET $A = [v_1 \cdots v_n]$. THEN A IS INVERTIBLE, SO $\{v_1, \dots, v_n\}$ IS A BASIS FOR $\mathbb{R}^n$ BY EXAMPLE 3.

(26.) SINCE $\sin 2t = 2 \sin t \cos t$, THIS IS A LINEARLY DEPENDENT SET. A BASIS FOR SPAN $\{\sin t, \sin 2t, \sin t \cos t\}$ IS $\{\sin t, \sin 2t\}$.

4.3 : 2, 6, 10, 12, 14, 34

(2.) THIS SET IS NOT A BASIS FOR $\mathbb{R}^3$ FOR TWO REASONS: IT IS LINEARLY DEPENDENT (BECAUSE IT CONTAINS THE ZERO VECTOR) AND IT DOES NOT SPAN $\mathbb{R}^3$ (BECAUSE SPAN $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ IS THE PLANE $z = x$ IN $\mathbb{R}^3$.)

(6.) THIS SET IS NOT A BASIS FOR $\mathbb{R}^3$ BECAUSE IT DOES NOT SPAN $\mathbb{R}^3$. IT IS, HOWEVER, LINEARLY INDEPENDENT.

(10.) $\begin{bmatrix} 1 & 0 & -5 & 1 & 4 & 0 \\ -2 & 1 & 6 & -2 & -2 & 0 \\ 0 & 2 & -8 & 1 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 1 & 4 & 0 \\ 0 & 1 & -4 & 0 & 6 & 0 \\ 0 & 2 & -8 & 1 & 9 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -5 & 1 & 4 & 0 \\ 0 & 1 & -4 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 & 7 & 0 \\ 0 & 1 & -4 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \end{bmatrix}$

BASIC VARIABLES: $x_1, x_2, x_4$

FREE VARIABLES: $x_3, x_5$

$$\begin{cases} x_1 = 5x_3 - 7x_5 \\ x_2 = 4x_3 - 6x_5 \\ x_4 = 3x_5 \end{cases} \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5x_3 - 7x_5 \\ 4x_3 - 6x_5 \\ x_3 \\ 3x_5 \\ x_5 \end{bmatrix}$$

$= x_3 \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ THUS $\left\{ \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$

IS A BASIS FOR NUL A.

(12.) $\left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right\}$

(14.) $A \sim B \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -\frac{7}{5} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{cases} x_1 = -2x_2 - 4x_4 \\ x_3 = \frac{7}{5}x_4 \\ x_5 = 0 \end{cases}$$

SO A BASIS FOR NUL A IS $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \\ 0 \end{bmatrix} \right\}$

BY THEOREM 6, A BASIS FOR COL A IS $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}$.

(34.) $p_1 + p_2 - 2p_3 = 0$

$\{1+t, 1-t\}$ IS A BASIS FOR SPAN $\{p_1, p_2, p_3\}$.