# MATH 22 LINEAR ALGEBRA FALL '04
# HOMEWORK #5 ANSWER KEY

3.1: 2, 6, 10, 14, 22, 30, 36

(2.) (a.) $\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix} = 0\begin{vmatrix} -3 & 0 \\ 4 & 1 \end{vmatrix} - 5\begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & -3 \\ 2 & 4 \end{vmatrix}$

$= 0(-3) - 5(4) + 1(22) = 2.$

(b.) $\begin{vmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{vmatrix} = -5\begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - 4\begin{vmatrix} 0 & 1 \\ 4 & 0 \end{vmatrix}$

$= -5(4) - 3(-2) - 4(-4) = 2.$

(6.) $\begin{vmatrix} 5 & -2 & 4 \\ 0 & 3 & -5 \\ 2 & -4 & 7 \end{vmatrix} = 5\begin{vmatrix} 3 & -5 \\ -4 & 7 \end{vmatrix} + 2\begin{vmatrix} 0 & -5 \\ 2 & 7 \end{vmatrix} + 4\begin{vmatrix} 0 & 3 \\ 2 & -4 \end{vmatrix}$

$= 5(1) + 2(10) + 4(-6) = 1.$

(10.) $\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{vmatrix} = -3\begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix}$

$= -3\left(5\begin{vmatrix} -2 & 2 \\ -6 & 5 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 2 & -6 \end{vmatrix}\right)$

$= -3(5(2) + 4(-2)) = -6.$

(14.) $\det A = -3\begin{vmatrix} 3 & 2 & 4 & 0 \\ 0 & -4 & 1 & 0 \\ -5 & 6 & 7 & 1 \\ 2 & 3 & 2 & 0 \end{vmatrix}$

$= -3(-1)\begin{vmatrix} 3 & 2 & 4 \\ 0 & -4 & 1 \\ 2 & 3 & 2 \end{vmatrix} = 3\left(3\begin{vmatrix} -4 & 1 \\ 3 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 4 \\ -4 & 1 \end{vmatrix}\right)$

$= 3(3(-11) + 2(18)) = 9.$

(22.) REPLACEMENT. DOESN'T AFFECT DETERMINANT.
(SEE THEOREM 3, P. 192.)

(30.) $\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1.$

(36.) $\det EA = \det \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det \begin{bmatrix} a & b \\ ka+c & kb+d \end{bmatrix}$

$= a(kb+d) - b(ka+c) = kab + ad - kab - bc$

$= ad - bc = 1(ad-bc) = \det \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= (\det E)(\det A).$

3.2: 6, 12, 24, 32, 34, 42

(6.) $\begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix} = 1 \begin{vmatrix} 1 & 5 & -3 \\ 0 & -18 & 12 \\ 0 & 3 & -1 \end{vmatrix} = 1 \cdot 6 \begin{vmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 3 & -1 \end{vmatrix}$

$= 1 \cdot 6 \cdot 1 \begin{vmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 6(1)(-3)(1) = -18.$

CHECK: $\begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix} = 1 \begin{vmatrix} -3 & 3 \\ 13 & -7 \end{vmatrix} - 5 \begin{vmatrix} 3 & 3 \\ 2 & -7 \end{vmatrix} - 3 \begin{vmatrix} 3 & -3 \\ 2 & 13 \end{vmatrix}$

$= 1(-18) - 5(-27) - 3(45) = -18.$

(12.) $\begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 4 & 6 & 6 \\ \frac{3}{2} & 0 & 1 & 0 \end{vmatrix} = -6 \begin{vmatrix} -1 & 2 & 3 \\ 3 & 4 & 3 \\ \frac{3}{2} & 0 & 1 \end{vmatrix}$

$= -6 \left( \frac{3}{2} \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 3 & 4 \end{vmatrix} \right) = -6 \left( \frac{3}{2}(-6) - 10 \right) = 114.$

(24.) $\begin{vmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ -7 & 2 & 6 \end{vmatrix} = 11 \neq 0 \Rightarrow \begin{bmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ -7 & 2 & 6 \end{bmatrix}$ INVERTIBLE

$\Rightarrow \begin{bmatrix} 4 \\ 6 \\ -7 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}$ LINEARLY INDEPENDENT.

(32.) $\det(rA) = r^n \det A$, BY $n$ APPLICATIONS OF THEOREM 3c (P. 192.)

(34.) USING THEOREM 6 (P. 196) AND THEOREM 2 (P. 189),
$\det(PAP^{-1}) = (\det P)(\det A)(\det P^{-1})$
$= (\det P)(\det P^{-1})(\det A) = (\det PP^{-1})(\det A)$
$= (\det I)(\det A) = 1 \cdot \det(A) = \det(A)$.

(42.) $\det(A+B) = \det A + \det B$

$\Leftrightarrow \begin{vmatrix} a+1 & b \\ c & d+1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$\Leftrightarrow (a+1)(d+1) - bc = 1 + ad - bc$

$\Leftrightarrow ad + a + d + 1 - bc = 1 + ad - bc$

$\Leftrightarrow a + d = 0$.

4.1 : 2, 6, 8, 30

(2.) (a.) LET $u = \begin{bmatrix} x \\ y \end{bmatrix}$, $c \in \mathbb{R}$.

$u \in W \Rightarrow xy \geq 0 \Rightarrow c^2 xy \geq 0$ (SINCE $c^2 \geq 0$)

$\Rightarrow (cx)(cy) \geq 0 \Rightarrow cu = \begin{bmatrix} cx \\ cy \end{bmatrix} \in W.$

THUS W IS CLOSED UNDER SCALAR MULTIPLICATION.

(b.) $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W$, $v = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \in W$

$u + v = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \notin W.$

THUS W IS NOT CLOSED UNDER ADDITION, SO IT IS NOT A SUBSPACE OF $\mathbb{R}^2$.

(6.) THIS IS NOT A SUBSPACE OF $\mathbb{P}_n$ BECAUSE IT IS NOT CLOSED UNDER ADDITION OR SCALAR MULTIPLICATION, AND DOES NOT CONTAIN $\vec{0}$.

(8.) THIS IS A SUBSPACE OF $\mathbb{P}_n$ BECAUSE IT CONTAINS THE ZERO POLYNOMIAL, AND IS CLOSED UNDER ADDITION AND SCALAR MULTIPLICATION.

(30.) SUPPOSE $c \neq 0$. $cu = 0 \Rightarrow \frac{1}{c}(cu) = \frac{1}{c}(0) = 0$

$\Rightarrow (\frac{1}{c} \cdot c) u = 0 \Rightarrow 1 \cdot u = 0 \Rightarrow u = 0$

(USING AXIOMS 9 AND 10.)

4.1 : 12, 16, 22

(12.) $W = \text{SPAN}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix} \right\}$ WHICH IS A SUBSPACE OF $\mathbb{R}^4$

BY THEOREM 1 (P. 221.)

(16.) W IS NOT A VECTOR SPACE BECAUSE $\vec{0} \notin W$.
TO SEE THIS, SUPPOSE BY WAY OF CONTRADICTION THAT $\vec{0} \in W$. THEN THERE IS A SOLUTION TO THE SYSTEM

$$\begin{aligned} -a+1 &= 0 \\ a-6b &= 0 \\ 2b+a &= 0. \end{aligned}$$

SINCE $a=1$ BY THE FIRST EQUATION,
$1-6b=0$ AND $2b+1=0$,
THUS $b = \frac{1}{6} = -\frac{1}{2}$. THIS IS A CONTRADICTION, BECAUSE $\frac{1}{6} \neq -\frac{1}{2}$.

(22.) H IS A SUBSPACE OF $M_{2\times 4}$.

PROOF: THE $2\times 4$ ZERO MATRIX $\begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} = \vec{0}$ IS CONTAINED IN H.
LET $A, B \in H$, $c \in \mathbb{R}$.
$F(A+B) = FA + FB = \vec{0} + \vec{0} = \vec{0}$, THUS $A+B \in H$, SO H IS CLOSED UNDER ADDITION.
$F(cA) = cFA = c\vec{0} = \vec{0}$ SO H IS CLOSED UNDER SCALAR MULTIPLICATION. QED

4.2: 2, 6, 8, 10, 19, 22, 24

(2.) $w \in \text{Nul } A$ BECAUSE

$$Aw = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}.$$

(6.) WE SOLVE THE HOMOGENEOUS LINEAR SYSTEM $Ax = \vec{0}$.

$$\begin{bmatrix} 1 & 5 & -4 & -3 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 6 & -8 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

BASIC VARIABLES: $x_1, x_2$

FREE VARIABLES: $x_3, x_4, x_5$

$$\begin{cases} x_1 = -6x_3 + 8x_4 - x_5 \\ x_2 = 2x_3 - x_4 \end{cases}$$

$$\Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -6x_3 + 8x_4 - x_5 \\ 2x_3 - x_4 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

THUS $\text{Nul } A = \text{SPAN} \left\{ \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(8.) THIS IS NOT A VECTOR SPACE BECAUSE IT DOES NOT CONTAIN THE ZERO VECTOR.

(10.) THIS IS A VECTOR SPACE BECAUSE IT IS THE NULL SPACE OF $\begin{bmatrix} 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}$.

(18.) NUL A IS A SUBSPACE OF $\mathbb{R}^3$ $(k = 3)$.
COL A IS A SUBSPACE OF $\mathbb{R}^4$ $(k = 4)$.

(22.) THERE ARE INFINITELY MANY CORRECT ANSWERS.
FOR EXAMPLE, $\begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} \in \text{Nul } A$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \text{Col } A$.

(24.) $w \in$ COL A AND $w \in$ NUL A.
$w \in$ NUL A BECAUSE $Aw = \vec{0}$.
$w \in$ COL A IFF $w$ IS CONTAINED IN THE SPAN OF THE COLUMNS OF A:

$$\begin{bmatrix} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 24 & 6 & 27 & -6 \\ 24 & 16 & 32 & 4 \\ 24 & 0 & 24 & -12 \end{bmatrix}$$

$$\sim \begin{bmatrix} 24 & 6 & 27 & -6 \\ 0 & 10 & 5 & 10 \\ 0 & -6 & -3 & -6 \end{bmatrix} \sim \begin{bmatrix} 8 & 2 & 9 & -1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 8 & 2 & 9 & -1 \\ 0 & 0 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ CONSISTENT, THUS $w \in$ COL A.