Things to Know About the Inner Product

Axioms of the inner product

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$;
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$;
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$;
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ **iff** $\mathbf{u} = \mathbf{0}$.

Definitions

1. **Length:** $||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

2. **Distance:** $\text{dist}(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$.

2. $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Properties of the length

1. $||c\mathbf{v}|| = |c|\,||\mathbf{v}||$.
2. **Pythagoras Theorem:** Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal **iff**

$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2.$$

3. **Cauchy-Schwarz Inequality:** $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq ||\mathbf{u}||\,||\mathbf{v}||$.
4. **Triangle Inequality:** $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$.