1. I decide to build a die and roll it once. We let X be the random variable that is the side of the die, and the outcome space of possible rolls are $\Omega = \{1, 2, 4, 7, 14\}$. Moreover, the die is loaded so that the probabilities of rolling each side are assigned as follows $m_X(k) = P(X = k) = \frac{k}{28}$. Verify that this defines a distribution, i.e. $1 = \sum_{k \in \Omega} m_X(k)$, and find the expected value of a roll of this die.

2. Suppose that X is a random variable with $E(X) = 100$ and $V(X) = 15$. Find
   (a) $E(X^2)$.
   (b) $E(2X + 4)$.
   (c) $E(-X)$.
   (d) $V(-X)$.

3. Let X and Y be the random variables for two independent rolls of a fair die. Define the new variable $S = X + Y$, the sum of the rolls of the die. Write down the distribution function for the random variable, S, and calculate the expected value of S.

4. A family decides to have children until it has a girl or until it has four children, whichever comes first. Assume that each child is a girl with probability 1/2. Find the expected number of boys and the expected number of girls in this particular family.

5. A box contains n tickets labeled $1, 2, \ldots, n$. Three tickets are drawn at random *with replacement* from the box. If X denotes the minimum of the three ticket numbers, use the tail-sum formula to find the expectation of X.

   Tail-sum formula: For some random variables, it is difficult to calculate the expectation using the usual formula $E(X) = \sum_{x \in \Omega} xP(X = x)$. Sometimes this alternative calculation is useful. If X is a random variable taking values on any subset of the non-negative integers, we have

   $$E(X) = \sum_{k=1}^{\infty} P(X \geq k),$$

   and this is called the tail-sum formula. *Hint*: Notice that this sum will be finite in our case, because $P(X \geq k) = 0$ for all $k > n$.

6. A die is rolled twice. Let X denote the sum of the two numbers that turn up, and Y the difference of the numbers (specifically, the number on the first roll minus the number on the second). Show that $E(XY) = E(X)E(Y)$. Are X and Y independent random variables?

7. Exactly one of eight similar keys opens a door.

   (a) If you try the keys, one after another (never trying a key that you had already tried), what is the expected number of keys (including the success) that you will have to try before opening the door?

   (b) Now suppose you try the keys, say out of a box, but after you try each one, you put the key back in the box (potentially to be drawn again). What is the expected number of keys you will have to try before success?

   (c) In general, with n keys, what are the expected number of keys you must try for each method?

Something to ponder: Did you notice that the sides of the die in the first question are all of the divisors of 28? It's quite peculiar that if you take the divisors of 28 (not including 28 itself) and add them together, you get

$$1 + 2 + 4 + 7 + 14 = 28.$$

When else does this happen? Well, it happens for the number 6 because $1 + 2 + 3 = 6$. The numbers 496 and 8128 are the next two numbers with the property that the sum of the divisors is equal to the number itself; and we call numbers with this property *perfect*. These patterns were first realized by Euclid way back in 300 BC, and no doubt observed by many other people in the years since. Then we begin to ask: What numbers are perfect? How many are there? Are they all even?

The answer right now is that we don't have all of the answers yet. We know that all even perfect numbers are exactly those of the from $2^{p-1}(2^p - 1)$ where $2^p - 1$ is a prime number (called a Mersenne prime). But we don't know if there are infinitely many prime numbers of this form, and so we don't know (although we probably believe) that there are infinitely many perfect numbers. And what about the existence of an odd perfect number? This turns out to be a rather difficult question, and is something that has puzzled mathematicians for over 2000 years at this point!