Math 20
Homework 5
Due: July 29, 2015

Solve the following problems and explain your reasoning.

Book problems: 5.2.2, 6.2.22, 6.2.23

4. The aim of this problem is to show that the normal density $\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2}$ is a density function, i.e. that:

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}dx = 1.$$

After substituting $y = (x-\mu)/\sigma$, we see that:

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2/2}dy.$$

Thus it suffices to show $\int_{-\infty}^{\infty} e^{-y^2/2}dy = \sqrt{2\pi}$. For simplicity, set $I = \int_{-\infty}^{\infty} e^{-y^2/2}dy$. Notice that:

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-y^2/2}dy \int_{-\infty}^{\infty} e^{-x^2/2}dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}dydx. \end{aligned}$$

Evaluate this last expression by switching to polar coordinates and show that it is equal to 2π.

5. (a) Show that the normal density has variance σ^2.

(b) Suppose that X is a continuous random variable with normal density. Using Wolfram Alpha or Mathematica compute the probability that X is within one standard deviation of μ, i.e. $P(|X-\mu| \leq \sigma)$. What is the probability that X is within two standard deviations of μ?

(c) Using your answer to part (b) explain why you would be surprised if in an experiment with outcome modeled by a normal random variable, the outcome was further than two standard deviations from the expected value.