HW #2

3.1.3 | EACH LETTER OF THE WORD HAS 2 POSSIBILITIES, 0 OR 1.

32 LETTERS GIVES 2^{32} DIFFERENT WORDS.

3.1.4 | 38 DIFFERENT DEAD PRESIDENTS. THE PROB THAT THEY DIED ON DIFFERENT DAYS:

$$\frac{1}{365^{38}} \cdot (365)_{38} \approx 0.136$$

I WOULDN'T BET ON IT. THOMAS JEFFERSON, JOHN ADAMS, AND JAMES MONROE ALL DIED ON JULY 4th

3.1.8 | $\binom{n}{j}$ = # OF SUBSETS OF SIZE j. SO ALL POSSIBLE SUBSETS:

$$\sum_{j=0}^{n} \binom{n}{j} = (1+1)^n = 2^n$$

BY THE BINOMIAL THM.

3.1.10 | THERE ARE $52 \times 51 \times \ldots \times 40$ POSSIBILITIES FOR THE 13 CARDS.

THE # OF POSSIBLE 13 CARD HANDS WHERE THE LAST CARD IS ACE IS $4 \times 51 \times \ldots \times 40$. SO THE PROB OF GETTING AN ACE IS $\frac{4}{52} = \frac{1}{13}$.

3.1.23 | a) IN EITHER CASE THE PROB SHE GETS THE BEST CANDIDATE IS $1/n$.

b) SHE DEFINITELY PICKS THE BEST CANDIDATE IF THE SECOND BEST CANDIDATE IS IN THE FIRST HALF AND THE BEST CANDIDATE IS IN THE LATTER HALF.

THE PROB OF THIS OCCURING IS:

$$\frac{\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)(n-2)!}{n!} = \frac{n}{4(n-1)} > \frac{1}{4}.$$

SO P(SHE PICKS BEST CANDIDATE) $> \frac{1}{4}$

3.2.13 NUMBER OF SUBSETS OF SIZE n ARE $\binom{2n}{n}$. IF k IS BIGGER THAN n,

THEN THE # OF SUBSETS OF SIZE k ARE:

$$\binom{2n}{k} = \frac{2n(2n-1)\dots(2n-k+1)}{k!}$$

$r = k-n > 0$

$$= \binom{2n}{n} \underbrace{\frac{n(n-1)\dots(n-r+1)}{k(k-1)\dots(n+1)}}_{<1}$$

$$< \binom{2n}{n}.$$

FOR SMALLER k WE USE THE FACT THAT: $\binom{2n}{k} = \binom{2n}{2n-k}$. THAT IS:

IF $k < n$ THEN $\binom{2n}{k} = \binom{2n}{2n-k} < \binom{2n}{n}$ SINCE $2n-k > n$.

3.2.20 (a) $\frac{1}{\binom{52}{6}} \binom{13}{6}$ → NUMBER OF WAYS OF CHOOSING 6 CARDS FROM 13 HEARTS.

(b) $\frac{1}{\binom{52}{6}} \underbrace{\binom{4}{3}}_{\text{3 ACES}} \overbrace{\binom{4}{2}}^{\text{2 KINGS}} \underbrace{\binom{4}{1}}_{\text{1 QUEEN}}$

(c) $\frac{1}{\binom{52}{6}} \underbrace{\binom{13}{3}}_{\text{\# of ways of choosing 3 cards from 13.}} \binom{13}{3} \cdot \overset{\text{\# of choices of first suit}}{4} \cdot \underset{\text{\# of choices of 2nd suit.}}{3}$

3.2.36 (a) THERE IS A CORRESPONDENCE BETWEEN j-TUPLES OF POSITIVE INTEGERS SUMMING TO $\leq n$ AND CHOICES OF PLACEMENTS OF j DASHES | BETWEEN n DOTS •S. FOR INSTANCE $(1,3,1)$ GOES TO:

$$\bullet \mid \bullet \ \bullet \ \bullet \mid \bullet \mid$$

IN GENERAL LET OUR j-TUPLE BE $(n_1, n_2, \ldots, n_j)$. THEN WE PUT THE FIRST DASH TO THE RIGHT OF THE n_1^{th} DOT. WE PUT THE k^{th} DASH TO RIGHT OF THE $n_1+n_2+\ldots+n_k{}^{th}$ DOT. GIVEN A SEQUENCE OF DOTS AND DASHES ITS CLEAR WE OBTAIN n_k FROM COUNTING THE NUMBER OF DOTS BETWEEN THE k^{th} AND $k-1^{ST}$ DASH. THUS WE UNIQUELY OBTAIN A A j-TUPLE FROM A SEQUENCE AND VICE-VERSA.

THE NUMBER OF WAYS OF PLACING j DASHES AMONG n DOTS IS $\binom{n}{j}$.

THUS THE SAMPLE SPACE HAS SIZE $\binom{n}{j}$.

(b) LET $A_i = \{$ j-TUPLES w/ A 1 IN i^{th} SLOT $\}$. THEN WHAT WE WANT TO COMPUTE IS:

$$P(A_1 \cup A_2 \cup \ldots \cup A_j) = \sum_{k=1}^{j} P(A_k) - \sum_{1 \leq k < l \leq j} P(A_k \cap A_l) + \ldots$$

$$\text{NOW} \quad P(A_k) = \frac{\binom{n-1}{j-1}}{\binom{n}{j}}, \quad P(A_i \cap A_k \ldots \cap A_l) = \frac{\binom{n-l}{j-l}}{\binom{n}{j}}$$

$$\text{SO} \quad P(A_1 \cup \ldots \cup A_j) = j\,\frac{\binom{n-1}{j-1}}{\binom{n}{j}} - \binom{j}{2}\frac{\binom{n-2}{j-2}}{\binom{n}{j}} + \ldots + (-1)^{j-1}\frac{1}{\binom{n}{j}}$$

10) a) $a_n = n$ AND $b_n = n + \sqrt{n}$ GIVES TWO ASYMPTOTICALLY EQUIV SEQUENCES

SINCE $\frac{n+\sqrt{n}}{n} \to 1$ (BY L'HOPITAL'S RULE)

HOWEVER $\lim_{n\to\infty} |a_n - b_n| = \lim_{n\to\infty} \sqrt{n} \to \infty$

(b) SINCE $\lim_{n\to\infty} b_n \neq 0$ WE HAVE:

$$\lim_{n\to\infty} \left| \frac{a_n - b_n}{b_n} \right| = \left| \lim_{n\to\infty} \frac{a_n - b_n}{b_n} \right| = \left| \lim_{n\to\infty} \frac{a_n}{b_n} - \lim_{n\to\infty} \frac{b_n}{b_n} \right|$$

$$= |1 - 1| = 0$$

11) FIRST NOTICE

$$(1+i)^{4n} + (1-i)^{4n} = \sum_{k=0}^{n} \binom{4n}{k} \left(i^k + (-i)^k \right)$$

THIS IS ZERO PRECISELY WHEN k IS ODD. IT IS -2 WHEN k IS A MULTIPLE OF 2 AND 2 WHEN k IS A MULTIPLE OF 4

$$= 2\left[\binom{4n}{0} - \binom{4n}{2} + \binom{4n}{4} + \dots \right]$$

NOTICE THAT

$$(1+1)^{4n} + (1-1)^{4n} = \sum_{k=0}^{n} \binom{4n}{k} \left(1 + (-1)^k\right) = 2\left[\binom{4n}{0} + \binom{4n}{2} + \binom{4n}{4} + \dots \right].$$

ADDING BOTH SIDES OF THIS EQN TO THE EQN ABOVE GIVES:

$$2(-4)^{4n} + 2^{4n} = 2\left[\binom{4n}{0} - \binom{4n}{2} + \binom{4n}{4} + \dots \right] + 2\left[\binom{4n}{0} + \binom{4n}{2} + \dots \right]$$

$$= 4\left[\binom{4n}{0} + \binom{4n}{4} + \dots \right].$$

DIVIDING BOTH SIDES BY 4 GIVES THE DESIRED RESULT.