Mock Final Exam

Math 20

Fall 2014

Name: ____

Please read the following instructions before starting the exam:

- This exam is closed book. You may not give or receive any help during the exam, though you may ask the instructors for clarification if necessary.
- Calculators are allowed in the exam.
- Be sure to show all work whenever possible. Even if your final answer is incorrect, an appropriate amount of partial credit can be assigned.
- Please circle or otherwise indicate your final answer, if possible.
- The test has a total of ... questions, worth a total of ... points. Point values are indicated for each question.
- You will have three hours from the start of the exam to complete it.
- Good luck!

HONOR STATEMENT: I have neither given nor received help on this exam, and I attest that all the answers are my own work.

SIGNATURE:_______________________

DOES NOT APPLY TO PRACTICE EXAMS

1. **Multiple choice.** Circle the correct answer to each question. [5 points each, no partial credit].

I. Which of the following is true?

(a) If you toss a fair coin 8 times in a row, the sequence of results HHHHTTTT is more likely than HHHHHHHH.

(b) Any Markov chain has a unique stationary distribution.

(c) The variance of $Y = cX$, for some constant c, is $cV(X)$.

(d) It is not possible to have a uniform probability distribution over an infinite sample space.

II. Let X, Y, and Z be independent standard normal random variables. What is the standard deviation of $X + Y + Z$?

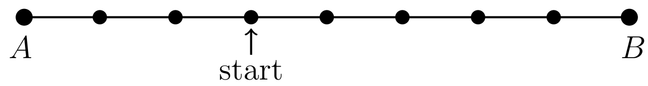
(a) 3

(b) 9

(c) $\sqrt{3}$

(d) 1

III. Alice and Bob made a wager, they're going to start a simple random walk on the graph pictured below at the indicated point. If it hits A before it hits B, Alice wins. Otherwise, Bob wins. What is the probability that Alice wins?



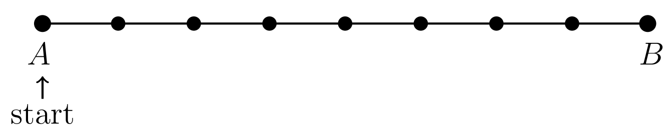
(a) 15

(b) 3/5

(c) 5/8

(d) 3/8

IV. Alice and Bob play a similar game as in part III, but this time they will start a simple random walk at A. Alice wins if the walk COMES BACK to A, after it leaves, before the first time it hits B. Bob wins otherwise. What is the probability that Alice will win?



(a) 1

(b) 1/8

(c) 1/15

(d) 7/8

2. You have a keyring of 6 keys, and they all look identical. One of them opens a lock. You're gonna try them one by one in the order they are arranged on a keyring. If all orders are equally likely, what is the expected number of keys you'll end up trying, until one finally opens the door? [10 points]

   Since any permutation of the keys is equally likely,the correct key has equal probability of being in position 1, 2, 3, 4, 5 and 6. So:

$$\mathbb{E}[X] = \frac{1}{6}1 + \frac{1}{6}2 + \frac{1}{6}3 + \frac{1}{6}4 + \frac{1}{6}5 + \frac{1}{6}6 = \frac{7}{2}.$$

3. Alice is preparing to take a qualifying exam (not multiple choice) in Probability. There is a list of 75 questions posted on the department's website. The exam consists of 5 of these questions, of which she must answer four correctly in order to pass. [20 points]

   (i) Alice goes into the exam knowing how to answer 65 of the questions. What is the expected number of questions, of the 5 she is asked, that she will be able to answer?

   (ii) Find the exact probability that Alice passes (you do not need to simplify your answer).

   (iii) Given that the exam contains one problem she doesn't know how to do, what is the probability that she will pass?

   (iv) Given that she passes the exam, what is the probability that the exam contained one problem she didn't know how to do?

   (v) Alice and Bob studied for this exam together. If you assume that they will definitely score within 1 point of each other, what is the probability that Bob will pass the exam, given that Alice passed?

(i)

$$\mathbb{E}[X] = 65 \times 5/75 = 65/15 = 13/3$$

(ii)

$$\frac{\binom{65}{5} + \binom{65}{4}\binom{10}{1}}{\binom{75}{5}}$$

(iii)

$$\frac{\binom{65}{4}\binom{10}{1}}{\binom{75}{5} - \binom{65}{5}}$$

(iv)

$$\frac{\binom{65}{4}\binom{10}{1}}{\binom{65}{5} + \binom{65}{4}\binom{10}{1}}$$

(v) That... seems like a strangely worded question. Pleaase ignore that for now. there won't be any sillyness lik this on the actual final.

4. Ted only eats dinner at Boloco or Molly's. However, he refuses to eat at Boloco two days in a row. If he eats at Molly's, there is an equal chance that he'll eat at Boloco or that he'll eat at Molly's the next day. Model this scenario using a Markov chain with two states, Boloco and Molly's. [20 points]

   (i) Write the transition matrix for this Markov chain.

   (ii) Find the stationary distribution for this Markov chain.

   (iii) If Ted just ate at Molly's, what is the expected number of days until he eats at Boloco?

   (iv) If Ted just ate at Boloco, what is the expected number of days until he eats at Boloco again?

(i)

$$P = \begin{array}{c} B \\ M \end{array} \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$$

(ii)

$$\begin{pmatrix} a & b \end{pmatrix} \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{pmatrix} a & b \end{pmatrix}$$

$$b/2 = a$$

$$a + b = 1$$

$$\begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} 1/3 & 2/3 \end{pmatrix}$$

(iii) Each day the probability is $1/2$. So the expected number of trials until first success is 2.

(iv) He will eat at Molly's the next day, and then have probability $1/2$ of eating at Bolloco until he finally does. So the expected number of days is 1+2=3.

5. Let X and Y be independent random variables distributed according to densities:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} \text{ if } 0 \leq x \\ 0 \text{ otherwise.} \end{cases}$$

$$f_Y(y) = \begin{cases} \nu e^{-\nu y} \text{ if } 0 \leq y \\ 0 \text{ otherwise.} \end{cases}$$

What is the probability density $f_Z(z)$ of $Z = X + Y$? [15 points]

If $z < 0$, $f_Z(z) = 0$. Otherwise, we have:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy$$

Now, neither y nor $z - y$ can be negative, so $0 \leq y \leq z$. We get:

$$f_Z(z) = \int_0^z f_X(z-y) f_Y(y) dy$$

$$f_Z(z) = \int_0^z \lambda e^{-\lambda(z-y)} \nu e^{-\nu y} dy$$

$$f_Z(z) = \lambda\nu \int_0^z e^{-\lambda z} e^{\lambda y} e^{-\nu y} dy$$

$$f_Z(z) = \lambda\nu e^{-\lambda z} \int_0^z e^{-(\nu-\lambda)y} dy = \frac{\lambda\nu}{\nu-\lambda} e^{-\lambda z} \int_0^z (\nu-\lambda) e^{-(\nu-\lambda)y} dy$$

Now, $\int_0^z (\nu-\lambda) e^{-(\nu-\lambda)y} dy$ is exponential density with parameter $\nu - \lambda$. Further,

$$\int_0^z (\nu-\lambda) e^{-(\nu-\lambda)y} dy = 1 - e^{-(\nu-\lambda)z},$$

and so, for $z \geq 0$:

$$f_Z(z) = \frac{\lambda\nu}{\nu-\lambda} e^{-\lambda z} (1 - e^{-(\nu-\lambda)z})$$

6. State the Law of Large Numbers (in any form) and use the Chebyshev's inequality to prove it. [15 points]

Law of Large Numbers: let S_n be the sum of n independent, identical random variables, each with mean μ and variance σ^2. Then for any $\epsilon > 0$:

$$\lim_{n\to\infty} P(|\frac{S_n}{n} - \mu| \geq \epsilon) = 0.$$

Chebyshev inequality: for any random variable X with mean m and variance $V(X)$,

$$P(|X - m| \geq \epsilon) \leq \frac{V(X)}{\epsilon^2}.$$

$$V(S_n) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}.$$

So,

$$\lim_{n\to\infty} V(S_n) = 0.$$

Then, in particular, since

$$0 \leq P(|\frac{S_n}{n} - \mu| \geq \epsilon) \leq \frac{V(S_n)}{\epsilon^2},$$

$$\lim_{n\to\infty} P(|\frac{S_n}{n} - \mu| \geq \epsilon) = 0.$$

$\square$

7. Using the Central Limit Theorem, estimate the probability that for S_{200} the sum of 200 consecutive rolls of a six-sided die, $675 \leq S_{200} \leq 725$. There is a normal distribution table at the end of this paper. [10 points]

For a single roll of a six-sided die, the mean is 3.5, and the variance is 35/12. Then:

$$\mathbb{E}[S_{200}] = 200 \times 3.5 = 700,$$

$$V(S_{200}) = 200 \times 35/12 = 1750/3$$

$$\sigma = \sqrt{\frac{1750}{3}} \simeq 24.15$$

$$\frac{25}{\sigma} \simeq 1.035$$

So according to the table,

$$P[675 \leq S_{200} \leq 725] \simeq 2 \times 0.4115 = 0.823.$$

8. Suppose that X is a continuous random variable picked from $[1/4, 3/4]$ uniformly at random. Y is a positive integer such that $Y < 1/X$, picked uniformly at random. For example, if $X \geq 1/2$, then Y is necessarily 1. If $1/2 \leq X \leq 1/3$, Y is either 1 or 2 with equal probability. Find the joint density $f_{X,Y}(x, y)$. [20 points]

Since X is dicked from $[1/4, 3/4]$ uniformly at random,

$$f_X(x) = \int_{\Omega_Y} f_{X,Y}(x,y)dy = \begin{cases} 2 \text{ if } x \in [1/4, 3/4] \\ 0 \text{ otherwise.} \end{cases}$$

If $X \geq 1/2$, then Y is necessarily 1. If $1/2 \leq X \leq 1/3$, Y is either 1 or 2 with equal probability. If $1/3 \leq X \leq 1/4$, Y is either 1, 2 or 3 with equal probability. So $\Omega_Y = \{1, 2, 3\}$, and:

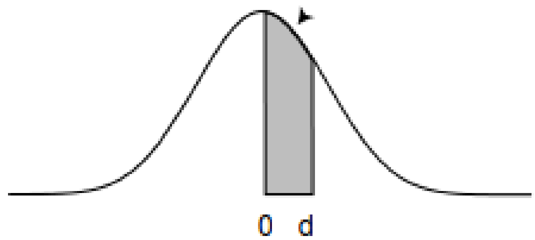
$$f_{X,Y}(x,1) = \begin{cases} 2 \text{ if } 3/4 \leq x \geq 1/2 \\ 1 \text{ if } 1/2 > x \geq 1/3 \\ 2/3 \text{ if } 1/3 > x \geq 1/4 \\ 0 \text{ otherwise.} \end{cases}$$

$$f_{X,Y}(x,2) = \begin{cases} 1 \text{ if } 1/2 > x \geq 1/3 \\ 2/3 \text{ if } 1/3 > x \geq 1/4 \\ 0 \text{ otherwise.} \end{cases}$$

$$f_{X,Y}(x,3) = \begin{cases} 2/3 \text{ if } 1/3 > x \geq 1/4 \\ 0 \text{ otherwise.} \end{cases}$$

Normal distribution table

NA (0,d) = area of shaded region



	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000