NAME: - ANSWER KEY -

Math 20

Summer 2015

Exam I

Instructions:

1. Write your name *legibly* on this page.
2. There are nine problems, some of which have multiple parts. Do all of them.
3. Explain what you are doing, and show your work. You will be *graded on your work*, not just on your answer. Make it clear and legible so I can follow it.
4. It is okay to leave your answers unsimplified. That is, if your answer is the sum or product of 5 numbers, you do not need to add or multiply them. Answers left in terms of binomial coefficients or factorials are also acceptable. However, do not leave any infinite sums or products, or sums or products of a variable number of terms.
5. There are a few pages of scratch paper at the end of the exam. I *will not look* at these pages unless you write on a problem "Continued on page..."
6. This exam is closed book. You may not use notes, calculators, or any other external resource. It is a violation of the honor code to give or receive help on this exam.

1. (10 points.) Show that $P(A \cap B) \geq 1 - P(\tilde{A}) - P(\tilde{B})$.

FROM THE FORMULA:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$

WE HAVE:

$$\begin{aligned} P(A \cap B) &= -P(A \cup B) + P(A) + P(B) \\ &\geq -1 + P(A) + P(B) \\ &= -1 + 1 - P(\tilde{A}) + 1 - P(\tilde{B}) \\ &= 1 - P(\tilde{A}) - P(\tilde{B}) \quad \checkmark \end{aligned}$$

2. (10 points.) If 5 married couples are seated at random in a row, compute the probability that no couple sits next to one another.

10 PEOPLE ARE SEATED IN A ROW. THERE ARE $10!$ WAYS OF ARRANGING THESE PEOPLE. LET A_i BE THE EVENT THAT COUPLE i IS SITTING NEXT TO ONE ANOTHER. WE NEED TO COMPUTE $1-P(A_1 \cup \dots \cup A_5)$.

WE USE THE LAW OF INCLUSION/EXCLUSION.

$$P(A_i) = \frac{1}{(10)!}\left[\begin{array}{l}\text{\# OF WAYS OF SITTING} \\ \text{A COUPLE TOGETHER}\end{array}\right]$$

$$= \frac{1}{10!}\left[\underbrace{2}_{\substack{\text{ARRANGEMENTS} \\ \text{OF THE 2} \\ \text{PEOPLE}}} \times \overbrace{9}^{\text{WHICH 2 CONSECUTIVE SLOTS TO PUT THE COUPLE}} \times \underbrace{(8!)}_{\substack{\text{ARRANGEMENTS OF} \\ \text{THE 8 OTHER PEOPLE}}}\right]$$

$$P(A_i \cap A_j) = \frac{1}{10!}\left[\underbrace{2^2}_{\substack{\text{ARRANGEMENTS} \\ \text{OF 4 PEOPLE}}} \times \overbrace{(9 \times 8)}^{\substack{\text{WHICH 2 CONSECUTIVE SLOTS TO PUT THE} \\ \text{2 COUPLES}}} \times \underbrace{(6!)}_{\substack{\text{ARRANGEMENTS OF} \\ \text{6 OTHER PEOPLE}}}\right]$$

$$P(\underbrace{A_i \cap A_j \cap \dots \cap A_k}_{k\text{ - COUPLES}}) = \frac{1}{10!}\left[2^k \times (9)_k \,(10-2k)!\right]$$

$$\text{SO } P(A_1 \cup A_2 \cup \dots \cup A_5) = \sum_{k=1}^{5} (-1)^{k-1} \frac{1}{10!}\left[2^k (9)_k (10-2k)!\right]\binom{5}{k}$$

AND THE ANSWER IS $1 - P(A_1 \cup \dots \cup A_5)$.

3. (10 points.) There are three urns labeled A, B, and C. Urn A contains 2 white and 4 red balls. Urn B contains 8 white and 4 red balls. Urn C contains 1 white and 3 red balls. Suppose one ball is chosen from each urn. What is the probability that the ball from urn A is white given that exactly 2 white balls were selected?

THE 3 POSSIBLE EVENTS ARE WRW, WWR, RWW.

WE WANT TO COMPUTE $P(A \to W \mid 2Ws) = \dfrac{P(A \to W \cap 2Ws)}{P(2Ws)}$.

THE PROB. OF 2Ws IS $P(WRW) + P(WWR) + P(RWW) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{3}{4} + \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{4}$

$P(A \to W \cap 2Ws) = P(WRW) + P(WWR)$ SO

$$P(A \to W \mid 2Ws) = \frac{P(WRW) + P(WWR)}{P(WRW) + P(WWR) + P(RWW)}.$$

4. (10 points.) Suppose you are throwing darts at a dart board. You are so bad at this that we might as well model it as picking a point (x, y) at random from inside the unit circle $\Omega = \{(x,y);\ x^2 + y^2 \leq 1\}$. Let $Z = x^2 + y^2$ the distance squared from the bullseye.

(a) Compute the cumulative distribution function for Z.

(b) Compute the density function for Z.

(a) $F(r) = P(Z \leq r) = \dfrac{\text{AREA OF CIRCLE OF RADIUS } r}{\text{AREA OF CIRCLE OF RADIUS } 1} = \dfrac{\pi r^2}{\pi} = r^2$

(b) THE DENSITY FUNCTION IS $\frac{d}{dr} F(r) = 2r$

5. (10 points.) Let $a_n \sim b_n$ denote that two sequences are *asymptotically equivalent.* Let a_n, b_n, and c_n be three sequences that do not limit to zero. Prove the following:

   (a) If $a_n \sim b_n$ then $b_n \sim a_n$.

   (b) If $a_n \sim b_n$ and $b_n \sim c_n$, then $a_n \sim c_n$.

(a) WE HAVE $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. SO $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1}{\lim_{n\to\infty} \frac{a_n}{b_n}} = \frac{1}{1} = 1$

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(b) $\lim_{n\to\infty} \frac{a_n}{c_n} = \lim_{n\to\infty} \frac{a_n}{b_n} \cdot \frac{b_n}{c_n}$

$= \lim_{n\to\infty} \frac{a_n}{b_n} \cdot \lim_{n\to\infty} \frac{b_n}{c_n}$

$= 1 \cdot 1$ (SINCE $a_n \sim b_n$ AND $b_n \sim c_n$)

6. (10 points.) Compute the cumulative distribution function for the following ~~random variables:~~ DENSITY FUNCTIONS

(a) $f(x) = \lambda e^{-\lambda x}$, $x \in (0, \infty)$

(b) $g(x) = 1/a$, $x \in (0, a)$

(c) $h(x) = 1/x^2$, $x \in (1, \infty)$.

(a) $$F(z) = \int_0^z \lambda e^{-\lambda t}\, dt = -e^{-\lambda t} \Big|_0^z = -e^{-\lambda z} + 1$$

(b) $$G(z) = \int_0^z \frac{1}{a}\, dt = \frac{z}{a}$$

(c) $$H(z) = \int_1^z \frac{1}{t^2}\, dt = -t^{-1} \Big|_1^z = \frac{-1}{z} + 1$$

7. (10 points.) Write:

$$\sum_{k=0}^{2n} \binom{4n}{2k}$$

without using $\sum$ or ....

(1) $(1+1)^{4n} = (2)^{4n} \overset{\text{BY THE BINOMIAL THM}}{=} \binom{4n}{0} + \binom{4n}{1} + \dots + \binom{4n}{4n}$

(2) $0 = (1-1)^{4n} = \binom{4n}{0} - \binom{4n}{1} + \binom{4n}{2} - \binom{4n}{3} + \dots + \binom{4n}{4n}$

ADDING LINES (1) + (2):

$$2^{4n} = 2\left[\binom{4n}{0} + \binom{4n}{2} + \binom{4n}{4} + \dots + \binom{4n}{4n}\right]$$

$$= 2 \sum_{k=0}^{2n} \binom{4n}{2k}$$

SO $\displaystyle\sum_{k=0}^{2n} \binom{4n}{2k} = \left(\frac{1}{2}\right) 2^{4n}$

8. (10 points.) A poker hand consists of 5 cards being dealt to you at random from a full deck. Find the probability of

   (a) a *flush* (5 cards of the same suit);

   (b) one pair (i.e. a hand of the form a, a, b, c, d with a, b, c, d distinct ranks).

(a) THE TOTAL NUMBER OF 5 CARD POKER HANDS IS $\binom{52}{5}$. THE NUMBER OF WAYS OF GETTING 5 CARDS OF THE SAME SUIT ARE (NUMBER OF SUITS) × (NUMBER OF WAYS OF PICKING 5 CARDS FROM 13)

SO THE PROBABILITY IS: $\frac{1}{\binom{52}{5}} \times 4 \times \binom{13}{5}$

(b) THE NUMBER OF WAYS OF GETTING 2 PAIRS IS

$$13 \times \binom{12}{3} \times \binom{4}{2} \times 4^3$$

13	$\binom{12}{3}$	$\binom{4}{2}$	4^3
# OF WAYS OF PICKING THE a rank	# OF WAYS OF PICKING b, c, d RANKS	# OF DIFFERENT SUIT COMBOS FOR a	# OF WAYS OF PICKING SUITS FOR b, c, d.

SO THE PROBABILITY IS: $\frac{4^3 \cdot 13 \cdot \binom{12}{3} \cdot \binom{4}{2}}{\binom{52}{5}}$