5. (5) If X and Y are any two random variables, then the *covariance* of X and Y is defined by $\mathrm{Cov}(X, Y) = E\big((X - E(X))(Y - E(Y))\big)$. What is $\mathrm{Cov}(X, X)$? Show that, if X and Y are independent, then $\mathrm{Cov}(X, Y) = 0$; and show, by an example, that we can have $\mathrm{Cov}(X, Y) = 0$ and X and Y not independent.

6. (5) We have two instruments that measure the distance between two points. The measurements given by the two instruments are random variables X_1 and X_2 that are independent with $E(X_1) = E(X_2) = \mu$, where μ is the true distance. From experience with these instruments, we know the values of the variances σ_1^2 and σ_2^2. These variances are not necessarily the same. From two measurements, we estimate μ by the weighted average $\overline{\mu} = \omega X_1 + (1 - \omega)X_2$. Here ω is chosen in $[0, 1]$ to minimize the variance of $\overline{\mu}$.

(a) What is $E(\overline{\mu})$?

(b) How should ω be chosen in $[0, 1]$ to minimize the variance of $\overline{\mu}$?

7. (10) For a sequence of Bernoulli trials, let X_1 be the number of trials until the first success. For $j \geq 2$, let X_j be the number of trials after the $(j-1)$st success until the jth success. It can be shown that X_1, X_2, $\cdots$ is an independent trials process.

(a) What is the common ditribution, expected value, and variance for X_j?

(b) Let $T_n = X_1 + X_2 + \ldots + X_n$. Then T_n is the time until the nth success. Find $E(T_n)$ and $V(T_n)$.

(c) Use the result of (b) to find the expected value and variance for the number of tosses of a coin until the nth occurance of a head.

9. (5) Suppose that n people have their hats returned at random. Let $X_i = 1$ if the ith person gets his or her own hat back and 0 otherwise. Let $S_n = \sum_{i=1}^{n} X_i$. Then S_n is the total number of people who get their own hats back. Show that

(a) $E(X_i^2) = 1/n$.

(b) $E(X_i, X_j) = 1/n(n-1)$ for $i \neq j$.

(c) $E(S_n^2) = 2$.

(d) $V(S_n) = 1$.

11. (5) Show that, if X and Y are random variables taking on only two values each, and if $E(XY) = E(X)E(Y)$, then X and Y are independent.