1. Consider flipping of two fair coins. Let

$$X = \begin{cases} 1 & \text{if first coin comes up heads} \\ 0 & \text{if first coin is tails} \end{cases}$$

Let

$$Y = \begin{cases} 1 & \text{if second coin comes up heads} \\ 2 & \text{if second coin comes up tails} \end{cases}$$

Let $Z = X + Y$. Find the probability distribution of Z. Find the $E(Z)$.

Z can take values $\{1+1, 0+1, 1+2, 0+2\} = \{1, 2, 3\}$

$P(Z=1) = P(X=0, Y=1) = P(X=0)P(Y=1) = \frac{1}{2}\cdot\frac{1}{2} = 1/4$ ↑ due to independence

$P(Z=2) = P(X=0)P(Y=2) + P(X=1)P(Y=1) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$

$P(Z=3) = P(X=1)P(Y=2) = \frac{1}{4}$

This gives the distribution Z. We can check $P(Z=2)+P(Z=3)+P(Z=1) = 1$

$E(Z) = 1\cdot\frac{1}{4} + 2\cdot\frac{1}{2} + 3\cdot\frac{1}{4} = \frac{1}{4} + 1 + \frac{3}{4} = 2$.

2. Let S_n be the number of heads in a fair coin toss. What is the limit as $n \to \infty$ of each of the following probabilities? Justify your answers.

(a) $P(n/2 - 100 < S_n < n/2 + 100)$

This is the same as problem 8 in section 8.1.

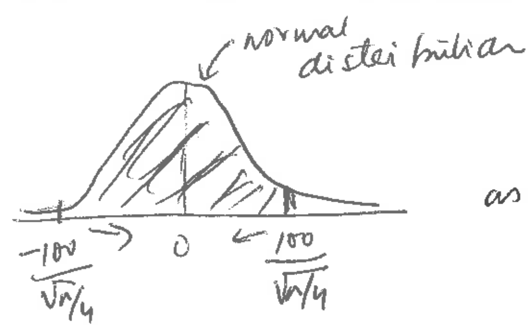
$$lim_{n\to\infty} P(n/2 - 100 < S_n < n/2 + 100) = 0.$$

See also a different solution done in detail in HW 5 write-up. Below is justification using CLT.
Note $E(S_n) = \frac{n}{2}$. And standard deviation of $S_n = \sqrt{\frac{n}{4}}$. Then by CLT (theorem 9.4 of text).

$$P(n/2 - 100 < S_n < n/2 + 100) = P(\frac{-100}{\sqrt{n/4}} < \frac{S_n - n/2}{\sqrt{n/4}} < \frac{100}{\sqrt{n/4}}) \approx \frac{1}{\sqrt{2\pi}}\int_{-100/\sqrt{n/4}}^{100/\sqrt{n/4}} e^{-x^2/2}\,dx$$

This is the probability of the event shown in the figure. The area $\to 0$ as $n \to \infty$.

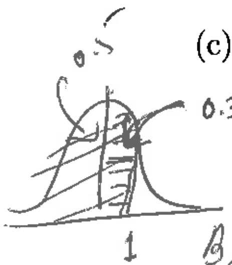
$$\lim_{n\to\infty} \frac{1}{\sqrt{2\pi}}\int_{-100/\sqrt{n/4}}^{100/\sqrt{n/4}} e^{-x^2/2}\,dx = 0$$



as $n \to \infty$ $\pm\frac{100}{\sqrt{n/4}} \to 0$

(b) $P(0.4n < S_n < 0.6n) =$ $P\left(0.4 < S_n/n < 0.6\right) = P\left(-0.1 < \frac{S_n}{n} - 0.5 < 0.1\right)$

$= P\left(\left|\frac{S_n}{n} - \frac{1}{2}\right| \leq 0.1\right) \longrightarrow 1$ by law of large numbers



(c) $P(S_n < 0.5n + 0.5\sqrt{n})$ $= P\left(\frac{S_n - 0.5n}{\sqrt{n/4}} < 1\right)$ Note $0.5\sqrt{n} = \sqrt{n/4}$.

↑ normalized random variable with mean $0.5n$ and std devn $\sqrt{n/4}$

By CLT, we can approx $P\left(\frac{S_n - 0.5n}{\sqrt{n/4}} < 1\right)$ by $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{1} e^{-x^2/2}dx = NA(1)$

$= 0.3413$
$+ 0.500$
$= 0.8413$

3. Give an example of a Markov Chain that is neither absorbing nor ergodic.

Solution: Problem 3 done in the practice problem handout for markov chains is neither absorbing nor ergodic. you can make up simpler ones by choosing 2 states to move to each other with some probability but not to any of the other states.

4. A process moves on the integers 1, 2, 3, 4 and 5. It starts at 1 and, on each successive step, moves to an integer greater than its present position, moving with equal probability to each of the remaining larger integers. State 5 is an absorbing state. Find the expected number of steps to reach state 5. This is problem 9, 11.2

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left(\begin{array}{cccc|c} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array}$$

Where

$$Q = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$N = (I - Q)^{-1} = \begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore expected number of steps to reach stage 5 starting from step 1 = sum of first row of the matrix N which is 25/12

5. Problems 13, 15, 19 section 11.2

#13. If Smith bets \$1 & wins with prob 0.4, he has a total of 2 dollars. If he bets \$1 when he is in state 2, he wins \$1 again with prob 0.4 & ends up with 3 dollars, ie. he ends up in state 3.

Transition matrix is as follows, for Timid Strategy.

	1	2	3	4	5	6	7	0	8
1	0	0.4	0	0	0	0	0	0.6	0
2	0.6	0	0.4	0	0	0	0	0	0
3	0	0.6	0	0.4	0	0	0	0	0
4	0	0	0.6	0	0.4	0	0	0	0
5	0	0	0	0.6	0	0.4	0	0	0
6	0	0	0	0	0.6	0	0.4	0	0
7	0	0	0	0	0	0.6	0	0	0.4
0	0	0	0	0	0	0	0	1	0
8	0	0	0	0	0	0	0	0	1

(The upper-right block in columns 0, 8 is labelled R.)

Using Matlab I get $NR = B$, where $N = (I-Q)^{-1}$

$B =$

	0	8
1	0.98	0.02
2	0.95	0.051
3	0.90	.096
4	0.84	0.165
5	0.73	0.2677
6	0.58	0.42
7	0.3469	0.65

↑ column gives probability of winning \$8.

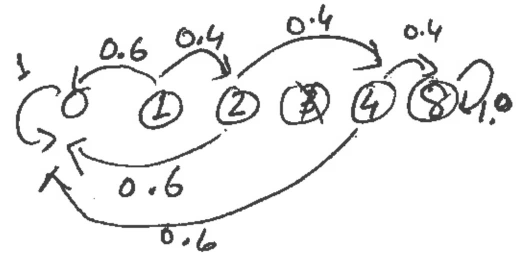
#13

(b) There can be multiple solutions to this problem. (Earlier soln had an error in R matrix)

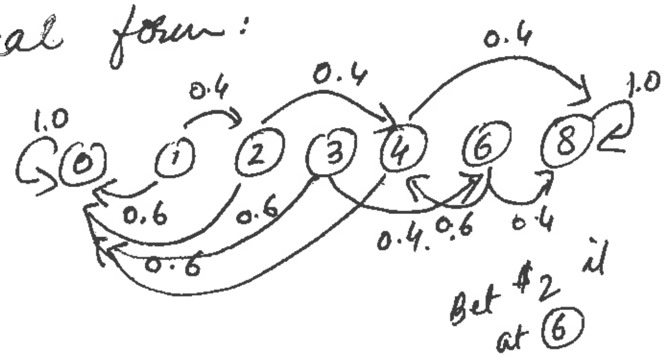
[A] If I assume he only bets amounts that when doubled, get him to \$8, then the amts that can be bet are \$1, 2, 4. Other states won't get him to \$8.

In this case Markov chain can be represented by the foll canonical form

	1	2	4	0	8
1	0	0.4	0	0.6	0
2	0	0	0.4	0.6	0
4	0	0	0	0.6	0.4
0	0	0	0	1	0
0	0	0	0	0	1



[B] If we assume that Smith always bets everything he has (except when he cannot) to break out of jail, we have the foll canonical form:



	1	2	3	4	6	0	8
1	0	0.4	0	0	0	0.6	0
2	0	0	0	0.4	0	0.6	0
3	0	0	0	0	0.4	0.6	0
4	0	0	0	0	0	0.6	0.4
6	0	0	0	0.6	0	0	0.4
0	0	0	0	0		1	0
8	0	0	0	0		0	1

4(a)

In the case of (A) form $(I-Q)^{-1} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0.4 & 0 \\ 0 & 0 & 0.4 \\ 0 & 0 & 0.6 \end{bmatrix} \right)^{-1}$

$$= \begin{pmatrix} 1 & -0.4 & 0 \\ 0 & 1.0 & -0.4 \\ 0 & 0 & 0.4 \end{pmatrix}^{-1} = \begin{bmatrix} 1 & 0.4 & 0.4 \\ 0 & 1.0 & 1.0 \\ 0 & 0 & 2.5 \end{bmatrix}$$

Matrix $B = NR$ gives probability of absorption

$$B = \begin{array}{c} \begin{matrix} 0 & 8 \end{matrix} \\ \begin{bmatrix} 1.08 & 0.16 \\ 1.20 & 0.40 \\ 1.50 & 1.00 \end{bmatrix} \end{array}$$

There is clearly something wrong here as probability of absorption into state 0 or state 8 should add to 1, and cannot exceed 1 for absorption into any state.

So let's look at [B].

Here $N = (I-Q)^{-1} = \begin{bmatrix} 1 & 0.4 & 0 & 0.16 & 0 \\ 0 & 1.0 & 0 & 0.40 & 0 \\ 0 & 0 & 1.0 & 0.24 & 0.40 \\ 0 & 0 & 0 & 1.000 & 0 \\ 0 & 0 & 0 & 0.6 & 1.00 \end{bmatrix}$ $R = \begin{bmatrix} 0.6 & 0 \\ 0.6 & 0 \\ 0.6 & 0 \\ 0.6 & 0.4 \\ 0 & 0.4 \end{bmatrix}$

$$B = N * R = \begin{array}{c} \begin{matrix} 0 & 8 \end{matrix} \\ \begin{bmatrix} 0.94 & 0.0640 \\ 0.84 & 0.1600 \\ 0.744 & 0.2560 \\ 0.60 & 0.400 \\ 0.36 & 0.64 \end{bmatrix} \end{array}$$

Clearly Bold strategy is better as prob of getting out of jail starting with \$3 is 0.2560, Vs 0.096 with timid strategy

(Note other solns are possible if you don't bet your entire amt, but can bet smaller amts than your holdings) 4(b)

11.2 #15 In the game of tennis if player A has probability of winning = 0.60 and B has prob of winning = 0.40 for any point played, then transition matrix in canonical form is.

$$\begin{array}{c|ccc|cc} & 3 & 4 & 5 & 1 & 2 \\ \hline 3 & 0 & 0.4 & 0 & 0.6 & 0 \\ 4 & 0.6 & 0 & 0.4 & 0 & 0 \\ 5 & 0 & 0.6 & 0 & 0.4 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{array}$$

(upper right block: R)

$$Q = \begin{pmatrix} 0 & 0.4 & 0 \\ 0.6 & 0 & 0.4 \\ 0 & 0.6 & 0 \end{pmatrix}$$

$$(I-Q)^{-1} = N = \begin{pmatrix} 1.46 & 0.77 & 0.31 \\ 1.15 & 1.923 & 0.77 \\ .69 & 1.154 & 1.46 \end{pmatrix}$$

If your book gives these probabilities as $1/3$ and $2/3$ instead, you can replace 0.4 with $1/3$ and 0.6 with $2/3$.

(a) $t = Nc = N\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2.54\\3.85\\3.31\end{pmatrix}$

Interpret this as:
time spent in Advantage A = 2.54
" " " Deuce = 3.85
" " " Adv. B = 3.31

(b) Absorption probability is given by

$$B = NR = \begin{array}{c} \\ 3 \\ 4 \\ 5 \end{array}\begin{pmatrix} \text{A wins} & \text{B wins} \\ 0.8769 & 0.1231 \\ 0.6923 & 0.3077 \\ 0.4154 & 0.5846 \end{pmatrix}$$

← deuce (row 3)

At deuce expected duration of game is 3.85 and prob. that B wins is 0.1231

6. Problems 10, 11, 16 section 11.3

#10. Example 11.10 has transition matrix (see pg 411 of text)

$$P = \begin{array}{c} \\ GG \\ Gg \\ gg \end{array} \begin{array}{c} \begin{array}{ccc} GG & Gg & gg \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ .5 & .5 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

GG is an absorbing state and its not possible to go to Gg or gg from GG. So this markov chain is <u>not</u> ergodic.

#11 Example 11.11 is again not ergodic. Not possible to get from (GG, GG) to any other state. (See pg 412).

#16 Eg 11.9 has $P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{pmatrix}$

Solve $wP = w$ to get $w = (1/4, 1/2, 1/4)$ the fixed vector.

(This was done in class)

7. A Restaurant feeds 300 customers. On the average 20% of the customers order steak.

   (a) Give a range for the number of steaks ordered on any given day so that you can be 95% percent sure that the actual number will fall in this range.

   (b) How many customers should the Restaurant have, on the average to be at least 95% sure that the number of customers ordering steak on that day falls in the 19% to 21% range.

Solution:

Apply CLT for Bernoulli trials here. Each order can be considered as a random variable X_i and $E(X_i) = 0.2$. $V(X_i) = (0.2 \times 0.8) = 0.16$.
Therefore expected value of $\sum_{i=1}^{300} X_i = np = 60$. Therefore expected number of steaks ordered = 60.Variance of steaks ordered = $0.16 * 300 = 48$ and one standard deviation = $\sqrt{48} = 6.93$. A 95% confidence limit corresponds to 2 standard deviations from the mean. So steaks are ordered in the range ~~$(60-7, 60+7) = (53, 67)$.~~ $(60-14, 60+14)$ $=(46,74)$

We need to find the value of n customers so that number of steaks ordered must lie in a 95% confidence level of length 0.01. Using the method of example 9.4 in text we want two standard deviations of the steak orders to be less than 1%. That is we want $2\sqrt{\frac{0.16}{n}} = 0.8/\sqrt{n} \leq 0.01$. Solving we get $n = 6400$. This is the number of customers the restaurant must have for a 95% confidence that the number of steaks ordered will be between 19% and 21%.

8. Prove the weak law of large numbers using the central limit theorem.

Solution Let $X_i, i = 1, 2, \dots, n$ be independent identically distributed random variables with $\mu = E(X_i)$ and $\sigma^2 = V(X_i)$ for each i. Let $S_n = \sum_{i=1}^{n} X_i$. We need to show that for every $\epsilon > 0$, $P(|\frac{S_n}{n} - \mu| \geq \epsilon) \to 0$ as $n \to \infty$. We assume that the CLT theorem holds. This means we can look at the area under the normal curve as an approximation of the probability of the event:

$$\{X_i : |\frac{S_n}{n} - \mu| \geq \epsilon\} = \{X_i : |S_n - n\mu| \geq n\epsilon\}$$

Now

$$lim_{n\to\infty} P(|\frac{Sn - n\mu}{\sqrt{n\sigma^2}}| \geq \frac{n\epsilon}{\sqrt{n\sigma^2}}) = 0$$

That is the area to the right of $\frac{\sqrt{n}\epsilon}{\sigma} \to 0$ as $n \to \infty$. We have shown that for every $\epsilon > 0$, $P(|\frac{S_n}{n} - \mu| \geq \epsilon) \to 0$ as $n \to \infty$, which is the result of the WLLN.

9. TRUE Or FALSE?

(a) The Central Limit Theorem (CLT) says that when you multiply a large number of random variables, the result will be a normal random variable.

False. CLT says nothing about multiplying random variables

(b) One of the assumptions of a Poisson ~~distribution~~ process is that the system has no memory

True. We can talk about the memory of a Poisson process and that is what is meant here. (Note it makes not sense to talk about memory of a probability distribution.) The Poisson process has the property that the number of successes of various intervals are independent. That is a Poisson process has no memory.

(c) The WLLN gives a limiting value of the sum of independent, identically distributed random variables

False.

(d) The CLT says that when you add a large number of random variables, the result will be a normal random variable.

False. The Theorem is about normalize sum of random variables. In addition the random variables have to satisfy some conditions such as being i.i.d for one.

(e) Chebyshev's inequality follows from the WLLN

False. The WLLN follows from the Chebyshev inequality.

(f) Chebyshev's inequality assumes that the random variable in questions has a symmetric distribution.

False. Chebyshev assumes nothing about the type of distribution.

Additionally please look over the practice problems of mid-term 1 and 2 - in particular the true/False.