MATH 20: DISCRETE PROBABILITY
SPRING 2017
MIDTERM II

Problem 1. TRUE OR FALSE

(1) Let $X_i, i = 1, \dots, n$ be random variables with some distribution. Let $S_n = \sum_{i=1}^{n} X_i$. Then $\frac{S_n}{n}$ is a random variable.

True

(2) Let $X_i, i = 1, \dots, n$ be independent random variables with some distribution, with finite expectation μ and finite variance σ. Let $S_n = \sum_{i=1}^{n} X_i$. The weak law of large numbers says that $\lim_{n\to\infty} \frac{S_n}{n} = \mu$.

False

(3) The weak law of large numbers implies that no matter what happens in the short run in the long run there will be a very high probability that the average of observations will be close to the expected value.

True

(4) A random variable X counts the number of successes in n Bernoulli trials, where the trials are done without replacement. Then X has a hypergeometric distribution.

True

(5) A random variable which has mean and variance λ has a Poisson distribution.

False

Problem 2. There are 5 yellow and 4 blue balls in an urn. 3 balls are drawn at random. What are the expected number of yellow balls drawn. If the first ball drawn is blue, find the expected number of yellow balls drawn.

Solution. Either $0, 1, 2$ or 3 yellow balls can be drawn. The total ways that 3 balls can be drawn is $\binom{9}{3} = 1/84$. Let $X =$ random variable representing number of yellow balls drawn. $E(X) = 0.P(X = 0) + 1.P(X = 1) + 2.P(X = 2) + 3.P(X = 3)$; $P(X = k) = \frac{\binom{5}{k}\binom{4}{(3-k)}}{84}$. Calculate $P(X = k)$ for $k = 0, 1, 2, 3$ and plug into the equation for $E(X)$.

$$E(X) = 0\frac{4}{84} + 1\frac{30}{84} + 2\frac{40}{84} + 3\frac{10}{84} = \frac{140}{84}$$

For the second part calculate:

$$E[X|\text{1st ball is blue}] = \sum_{k=0}^{2} k.P(X = k|\text{1st ball is blue}] = 0.\frac{\binom{5}{0}\binom{3}{2}}{\binom{8}{2}} + 1.\frac{\binom{5}{1}\binom{3}{1}}{\binom{8}{2}} + 2.\frac{\binom{5}{2}\binom{3}{0}}{\binom{8}{2}} = \frac{35}{28}.$$

One ball being fixed at blue there are a total of $\binom{8}{2} = 28$ ways of picking 2 balls from the remaining. The above expression is easy to simplify and you can try to do that.

Problem 3. Let X be the first time that a failure occurs in an infinite series of Bernoulli trials with probability p for success. What is $E(X)$? What is the expected number of tosses of a coin required to obtain the first tail?

Solution. We did this in class

Problem 4.

A number is chosen at random from the integers $1, 2, \ldots, n$. Let X be the number chosen. What is $V(X)$?

Solution. The question implies that the numbers are uniformly distributed with probability $1/n$ of any number being chosen. Let X be a random variable representing a number chosen from $1, \ldots, n$. $E(X) = n(n+1)/(2n) = (n+1)/2$; $E(X^2) = \sum_i^n i^2/n = \frac{n(n+1)(2n+1)}{6.n} = \frac{(n+1)(2n+1)}{6}$. Now write down the expression for $V(X) = E(X^2) - (EX)^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{(n+1)}{2}.\frac{(n-1)}{6}$;

Problem 5.

Asume that the probability that there is a significant accident in a nuclear power plant during one year is 0.001. If there are 50 nuclear power plants in a country what is the probability of at least one accident during a year?

Solution. If we think of an accident as a rare event, that is the probability of accident is small relative to a large number of power plants, then we can see that the number of accidents in a year to a close approximation will be Poisson random variable with parameter $\lambda = 0.001(50) = 0.05$. Let $X =$ the random variable denoting number of accidents in one year. Then $P(X = 0) = e^{-0.05}(0.05)^0/0! = e^{-0.05}$. Therefore $P(X \geq 1) = 1 - e^{-0.05}$

Problem 6. Let X and Y be two random variables defined on a finite sample space Ω. Assume that $X, Y, X+Y, X-Y$ all have the same distribution. Show that $P(X = 0) = 1$. What can you say about Y?

Solution. Since all the random variables have the same distribution,

$$E(X) = E(Y) = E(X - Y) = E(X) - E(Y) = E(X + Y) = E(X) + E(Y)$$

Equating the appropriate term we get $E(X) = 0$ and $E(Y) = 0$.
We also have $E[(X+Y)^2] = E[(X-Y)^2] = E(X^2) = E(Y^2)$. Now working with the second moments

$$E[(X+Y)^2] = E[X^2] + 2E[XY] + E[Y^2] = E[X^2] \implies 2E[XY] + E[Y^2] = 0$$

Using

$$E[(X-Y)^2] = E[X^2] - 2E[XY] + E[Y^2] = E[X^2]$$

we get $-2E[XY] + E[Y^2] = 0$. Hence $E[XY] = 0$ and repeating the above and equating with $E[Y^2]$ we can conclude the all second moments are zero. (You need to write down the expansion in detail to see this.) Then $V(X) = E[X^2] - E[X] = 0$. But $E(X) = 0 = E(Y)$. So X and Y are constant with probability one. And the only constant they can be is that prescribed by their expected value. Hence $P(X = 0) = 1 = P(Y = 0)$.

Problem 7. A communication system consists of n components, each of which functions independently with probability p. The total system will be able to operate effectively if atleast one-half of its components function. For what values of p is a 5-component system more likely to operate effectively that a 3 component system?

hint: write down the probability of each system being effective and compare them.

Solution. Let X be the binomial random variable representing the number of functioning components with parameters n, p.
The 5-component system will be effective if $X > 2.5$ and the 3 component system will be effective if $X > 1.5$
$P(X > 2.5) = \binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p) + p^5$ for the 5 component system.
$P(X > 1.5) = \binom{3}{2}p^2(1-p) + p^3$ for the 3-component system

5-component system is better if

$$10p^3(1-p)^2 + 5p^4(1-p) + p^5 > 3p^2(1-p) + p^3$$

The above reduces to $3(p-1)^2(2p-1) > 0 \implies p > \frac{1}{2}$. You should show the steps in between.

Problem 8. David is working on a math problem that he recognizes as having 3 possible approaches to the solution. The first approach will take him 2 hours to solve the problem. The second approach will take him 1 hour but he will be back to square one. The third approach will take him 1/2 hours with no result. Assume David is equally likely to choose any of the approaches each time he starts, what is the expected time for him to solve the problem?

Solution. Let X denote the time in hours till David solves the problem. Let Y be the random variable denoting the approach he takes to solve. Y takes value $1, 2, 3$ with equal probability.

$$\begin{aligned}E[X] &= E[X|Y=1]P[Y=1] + E[X|Y=2]P[Y=2] + E[X|Y=3]P[Y=3] \\ &= \frac{1}{3}(E[X|Y=1] + E[X|Y=2] + E[X|Y=3])\end{aligned}$$

Thus

$$E[X] = \frac{1}{3}(2 + 1 + E[X] + 0.5 + E[X])$$

Solve for $E[X]$. $E[X] = 3.5$.

Problem 9. Assume X takes integer values and is uniformly distributed over $[0, 10]$. Give an upper bound for $P(|X - E(X)| > 4)$.

Solution. X has a uniform distribution. $P(X = i) = \frac{1}{11}$

$$E(X) = \sum_{i=0}^{10} i.p(i) = \frac{1}{11}\frac{10.11}{2} = 5$$

$$V(X) = E[X^2] - E[X];\ E[X^2] = \sum_{i=0}^{10} i^2.p(i) = \frac{10.11.21}{6}.\frac{1}{11} = 35;\ So\ V(X) = 35 - 25 = 10$$

Upper bound using Chebyshev is $V(X)/4^2 = 10/16 = 5/8$

Problem 10. The number of TVs sold weekly at a store is a random variable with expected value 10. Give an upper bound on the probability that next week's sales exceed 18. Let the variance of number of TVs sold per week be 2. Give a lower bound for the probability that next week's sales will be between 8 and 12.

Solution. Let X be a random variable with value equal to number of Tv's sold per week. X is clearly non-negative. So we can use Markov inequalit. $P(X > 18) = P(X \geq 19) \leq 10/19$ Use chebyshev for second part to get $P(|X - 10| \geq 2) \leq \frac{2}{4}$. This is equivalent to saying $1 - P(8 < X < 12) \leq \frac{2}{4}$. Hence $P(8 < X < 12) > 1/2$.

Problem 11. Let X be a random variable with $E(X) = \mu$ and $V(X) = \sigma^2$. Show that for every positive integer k

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

This is an example in your text book.