MATH 20: DISCRETE PROBABILITY
SPRING 2017
PRACTICE PROBLEMS MIDTERM I

Problem 1. Mark True or False: No justification needed

(1) Let $\Omega = \{\omega_1, \omega_2, \ldots \omega_n\}$ be a sample space. Then it must be the case that $P(\Omega) = 1$
True

(2) The number of questions that you answer correctly on this review is an example of a discrete random variable. False (assuming you have studied) :)

(3) The expected value for a random variable must be a possible value of that random variable. False

(4) The probability of a student randomly guessing answers to a true/false exam is best modeled with a binomial distribution. True

(5) The formula for the binomial probability distribution takes into account both the probability of success as well as the probability of failure. True

(6) A probability distribution for a discrete variable depicts all possible mutually exclusive events, with the sum of the corresponding probabilities equalling 1.0. True

Problem 2. A coin comes up tails with probability p on any particular flip. Let the random variable X be the number of flips until, and including, the first time it comes up tails.

- What is the probability distribution of X?
- What is E(X)?

Solution. (~~This~~ 2nd part was done in class last on Wednesday - April 12, 2017)

$X = j$ if $(j-1)$ tosses are tails, followed by a heads.

sample space $\Omega = \{1, 2, 3, \ldots\}$, a countably infinite set.

Distribution of X is given by $m(x_j) = pq^{j-1}$ $\forall x_j \in \Omega$.

Here p = probability of tails. q = probability of Heads.

$p = 1 - q$.

clearly $m(x_j) \geq 0$ $\forall x_j \in \Omega$. Need to check $\sum_{i=1}^{\infty} m(x_i) = 1$

$$\sum_{j=1}^{\infty} m(x_j) = \sum_{i=0}^{\infty} pq^i = p\sum_{i=1}^{\infty} q^i = \frac{p}{1-q} = 1.$$

$$E(X) = \sum_{i=1}^{\infty} ipq^{i-1} = p\sum_{i=1}^{\infty} iq^{i-1}. \text{ Note } \sum_{i=1}^{\infty} iq^{i-1} = \frac{d}{dq}\sum_{i=0}^{\infty} q^i = \frac{1}{(1-q)^2}$$

$$\therefore E[X] = \frac{p}{(1-q)^2} = \frac{1}{(1-q)} = \frac{1}{p}$$

Problem 3. There are 21 balls in an urn, 6 of them are blue, 7 are red, and 8 are yellow. If you pick 5 balls from the urn at random, what is the probability that x of them will be blue, and y of them will be red, for any x, y?

Solution. $\binom{21}{5}$ ways to sample 5 balls from 21.

Clearly $x, y \in \{0,1,2,3,4,5\}$. And $0 \le x+y \le 5$.

Among 5 balls picked there are $\binom{6}{x}$ ways to pick blue balls.

" " " " " " $\binom{7}{y}$ " " " red balls.

" " " " " " $\binom{8}{5-(x+y)}$ ways to pick yellow balls.

$\therefore$ Probability of x blue, y red balls is $\binom{6}{x}\binom{7}{y}\binom{8}{5-(x+y)} \Big/ \binom{21}{5}$.

Problem 4.

- A sample space Ω conists of the ordered triples (i, j, k) where i, j, k are integers in $\{1, 2, 3\}$ and are either all different or all the same. List the members of Ω.
- Give the sample space Ω for a student chosen at random from a class of 10.

Solution. Ω has 9 elements (3 with all same & 3! with distinct i, j, k values)

$\Omega = \{(1,1,1), (2,2,2), (3,3,3), (1,2,3)(1,3,2), (2,1,3), (2,3,1), (3,1,2)(3,2,1)\}$

For second part.

$\Omega = \{1, 2, 3, \ldots, 10\}$. Where each outcome is the student number. That is the students are numbered $1, 2, \ldots, 10$.

Problem 5. What is Stirlings formula?

Solution. Sterling's formula is given as an approximation for $n!$:

$$n! \sim n^n e^{-n} \sqrt{2\pi n}$$

That is $n!$ is asymptotically equal to $n^n e^{-n} \sqrt{2\pi n}$

Problem 6. It has been discovered that 20% of major league baseball players use steroids. A certain drug test gives a positive for steroid use 99% of the time if you are using steroids, and 2% of the time if you are clean.

- (a) What is the probability of a random player testing positive?
- (b) What is the probability of being a steroid user if you test positive?

Solution.

Prior distribution is $P(S) = 0.20$. $P(NS) = 0.80$

$S =$ use of steroid. $NS =$ use of no steroid.

$E =$ positive drug test.

we have $P(E/S) = 0.99$, $P(E/NS) = 0.02$

(a) $P(E) = P(E/S)P(S) + P(E/NS)P(NS)$

$= (0.99)(0.20) + (0.02)\,0.80$

$= (1 - 0.01)(0.20) + .016 = .198 + 0.016 = 0.214$

(b) Find $P(S/E)$.

$$P(S/E) = \frac{P(E/S)\,P(S)}{P(E)}$$ (Since $P(S/E) = \frac{P(S \cap E)}{P(E)}$ and $P(E/S) = \frac{P(S \cap E)}{P(S)}$)

$$= \frac{0.99 \times 0.20}{0.214} = \frac{.198}{0.214} = \frac{99}{107}.$$

Problem 7. If $P(A) = \frac{1}{2}, P(B) = 1/4, P(C) = \frac{1}{8}. P(A \cup B) = \frac{3}{4}$ Find each of the following:

- $P(A \cap B^c)$
- $P(A^c \cap B^c)$
- A best estimate lower bound for $P(A \cup B \cup C)$ given the above information.

Solution. We use the inclusion-exclusion principle for $n = 3$. Recall the inclusion-exclusion relationship:

$$P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

$$\cdots + (-1)^{r+1} \textstyle\sum_{i_1<i_2<\ldots<i_r} P(A_{i_1} \cap A_{i_2} \ldots \cap A_{i_r}) \cdots$$

$$+(-1)^{n+1} P(\cap_{i=1}^{n} A_i)$$

$*$ For $n=3$: $P(A \cup B \cup C) = P(A)+P(B)+P(C) - P(A\cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

(1) $P(A \cap B^c) = P(A) + P(B^c) - P(A \cup B^c)$. $P(A)+P(B) = \frac{3}{4} = P(A \cup B) \Rightarrow A \cap B = \phi$

$A \cup B^c \supseteq A \Rightarrow P(A \cup B^c) = P(B^c)$

$\therefore P(A \cap B^c) = \frac{1}{2} + \left(1 - \frac{1}{4}\right) - \left(1 - \frac{1}{4}\right) = 1/2$.

(2) $P(A^c \cap B^c) = P(A^c) + P(B^c) - P(A^c \cup B^c)$ $\quad A^c \cup B^c = (A \cap B)^c = \Omega$

$= \frac{1}{2} + \frac{3}{4} - 1 = 1/4$.

Also directly by $A^c \cap B^c = (A \cup B)^c \Rightarrow P(A^c \cap B^c) = P((A\cap B)^c) = 1 - 3/4 = 1/4$

Use $*$ $P(A \cup B \cup C) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} - 0 - P(B \cap C) - P(A \cap C) + \underbrace{P(A \cap B \cap C)}$

$= 7/8 - P(B \cap C) - P(A \cap C) \geq 3/4$ as $B \cap C \subset C$ & $A \cap C \subset C$

Find the best estimate lower bound, given additionally $P(A \cap C) = \frac{1}{10}, P(B \cap C) = \frac{1}{12}$.

In this case

$P(A \cup B \cup C) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} - 0 - \frac{1}{10} - \frac{1}{12} + 0 = \frac{73}{120}$.

So this the best possible lower bound & upper bound as we can calculate $P(A \cup B \cup C)$ exactly.

Remark: There is an issue here $(B \cap C \cup A \cap C) \subset C$

But $P(B \cap C) + P(A \cap C) = \frac{1}{10} + \frac{1}{12} = \frac{11}{60} > \frac{1}{8}$!!

$P(B \cap C)$ & $P(A \cap C)$ should have been $1/20$ each.!!

Problem 8. John rolls 2 six-sided dice, and if the sum of the dice is divisible by 3, he wins \$6 dollars. If the sum is not divisible by 3, he loses \$3 dollars. What is John's expected winnings from playing this game?

Solution. Probability space Ω consists of 36 outcomes.
Ω_3 has 12 possible outcomes. These are $\{(1,2)(2,1),(3,3),(4,2)\ (2,4),(5,1)(1,5)$
$(4,5)(5,4)\ (6,3)(3,6)(6,6)\}$

$P(\Omega_3) = 1/3 \quad P(\Omega_3^c) = \frac{2}{3}$

$E[\text{winnings}) = 6 \cdot \frac{1}{3} - 3 \cdot \frac{2}{3} = 0.$

Problem 9. Refer to the hat check problem discussed in class.
In the hat check problem, assume that N people check in their hats and these are handed back randomly. Let $X_j = 1$ if the jth person gets her hat back. Otherwise $X_j = 0$. Find $E(X_j)$. Are X_j and X_k independent?

Solution.

$P(X_j) = \frac{(n-1)!}{n!} = \frac{1}{n}$. $E(X_j) = 1 \frac{1}{n} + 0 \cdot (1 - \frac{1}{n}) = \frac{1}{n}$ for any j.

If $E(X_j \cdot X_k) \neq E(X_j) E(X_k)$ for j, k then X_j, X_k are not independent.

$P(X_j X_k) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$ $\quad X_j X_k = \begin{cases} 1 & \text{if } j \,\&\, k \text{ get hat back} \\ 0 & \text{otherwise} \end{cases}$

$\therefore E(X_j X_k) = 1 \cdot \frac{1}{n(n-1)} + 0 \cdot \left(1 - \frac{1}{n(n-1)}\right) = \frac{1}{n(n-1)}$

Since $E(X_j) E(X_k) = \frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \neq \frac{1}{n(n-1)} = E(X_j \cdot X_k)$

X_j, X_k are not independent.

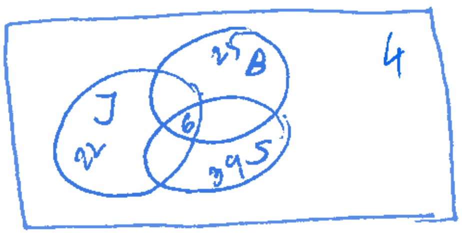
Problem 10. In a survey of the chewing gum tastes of a group of baseball players, it was found that:

- 22 liked juicy fruit J
- 25 liked spearmint S
- 39 like bubble gum B
- 9 like both spearmint and juicy fruit
- 17 liked juicy fruit and bubble gum
- 20 liked spearmint and bubble gum
- 6 liked all three
- 4 liked none of these

How many baseball players were surveyed?

J = juicy fruit, S = spearmint
B = bubble gum.

$|J| = 22$, $|S| = 25$, $|B| = 39$



$|J \cap S| = 9$.

$|J \cap S \cap B| = 6$

$|J \cap B| = 17$

$|(J \cap S \cap B)^c| = 4$.

$|S \cap B| = 20$

Using Inclusion - exclusion principal.

Total Surveyed Baseball players is given by (here $|\cdot|$ denotes # of elements in set.)

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap C \cap B| + |(A \cup B \cup C)^c|$$

Let $A = J$, $B = B$, $C = S$.

Total Surveyed $= 22 + 25 + 39 - 9 - 17 - 20 + 6 + 4 = 50$