1. (10 points.) Suppose you have an urn containing N red and M blue balls. Balls are selected randomly and replaced until a red ball is obtained.

   (a) What is the probability that n draws are needed (and no more)?

   (b) What is the expected number of draws until a red ball is obtained (here we do not count the last draw of a red ball as a draw)?

(a) THIS IS A SITUATION MODELED BY A GEOMETRIC DISTRIBUTION, SO

IF $T = \#$ OF DRAWS UNTIL A RED BALL. THEN

LAST BALL IS RED.

$$P(T=n) = \left(\frac{M}{N+M}\right)^{n-1}\left(\frac{N}{N+M}\right)$$

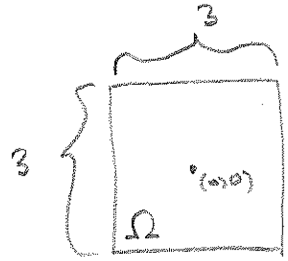
PROB OF NOT GETTING RED.

(b) $E(T)$ FOR A GEOMETRIC DIST. IS $\frac{1}{p}$ WHERE p = PROB OF PICKING A RED BALL:

$$E(T) = \left(\frac{N+M}{N}\right).$$

3. (10 points.) The county hospital is located at the center of a square whose sides are 3 miles wide. If an accident occurs within this square, then the hospital sends out and ambulance. The road network is rectangular, so the travel distance from the hospital, whose coordinates are $(0, 0)$, to the point (x, y) is $|x| + |y|$. If an accident occurs at a point that is uniformly distributed in the square, find the expected travel distance of the ambulance.

THE SAMPLE SPACE IS



$= [-1.5, 1.5] \times [-1.5, 1.5]$

W/ UNIFORM DISTRIBUTION $= \frac{1}{9}$.

$E(|X|+|Y|) = E(|X|) + E(|Y|)$

BY INDEPENDENCE

$$E(|X|) = \frac{1}{9}\int_{-1.5}^{1.5} |X| \, dx = \frac{2}{9}\int_0^{1.5} x \, dx = \frac{2}{9} \cdot \frac{1}{2} x^2 \Big|_0^{1.5}$$

$$= \frac{1}{9}\left(\frac{3}{2}\right)^2 = \frac{1}{4}$$

$E(|Y|)$ IS THE SAME. SO $E(|X|+|Y|) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

4. (10 points.) Derive the *One-sided Chebyshev* inequality which asserts: If X is a random variable with $E(X) = 0$ and variance $V(X) = \sigma^2$, then for any $a > 0$:

$$P(X \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}.$$

BY THE FORM OF THIS INEQUALITY WE SHOULD TRY TO APPLY MARKOV'S INEQUALITY SOMEHOW. NOW

$$P(X \geq a) = P(X + b \geq b + a)$$

$$\leq P\left((X+b)^2 \geq (a+b)^2\right)$$

BY MARKOV'S INEQUALITY

$$\leq \frac{E\left((X+b)^2\right)}{(a+b)^2}$$

$$= \frac{E(X^2 + 2bX + b^2)}{(a+b)^2}$$

$$= \frac{\sigma^2 + b^2}{(a+b)^2}.$$

SETTING $b = \frac{\sigma^2}{a^2}$ GIVES

$$\leq \frac{\sigma^2}{\sigma^2 + a^2}.$$

5. (10 points.)

(a) Let X be a random variable with mean 0 and variance 1. Find the smallest number n so that you can guarantee that $P(|X| \geq n) \leq 1/25$.

(b) Suppose that it is known that the number of cars produced in a Ford factory during a week is a random variable with mean 50. What can you say about the probability that a given week's production will exceed 75 cars? If the variance of a week's production is 25, what can you say about the probability that the production will be between 40 and 60 cars?

(a) USING CHEBYSHEV'S INEQUALITY WE HAVE:

$$P(|X| \geq n) \leq \frac{\sigma^2}{n^2}$$

WE WANT $\frac{\sigma^2}{n^2} = \frac{1}{n^2} \leq \frac{1}{25}$ SO $n \geq 5$

SO $n = 5$

(b) IN THE FIRST QUESTION IF WE DON'T HAVE ANY VARIANCE WE ONLY APPLY MARKOV'S INEQUALITY TO FIND:

$$P(X \geq 76) \leq \frac{50}{76}.$$

IF WE HAVE VARIANCE WE CAN APPLY CHEBYSHEV'S INEQUALITY TO FIND:

$$P(|X-50| \geq 11) \leq \frac{25}{(11)^2}$$

SO $P(|X-50| \leq 10) \geq 1 - \frac{25}{(11)^2}.$

6. (15 points.) Suppose that X is a continuous random variable with density function:

$$f_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{3} & \text{if } 0 \le t \le 3\,, \\ 0 & \text{if } t > 3 \end{cases}$$

and Y is a continuous random variable with density function:

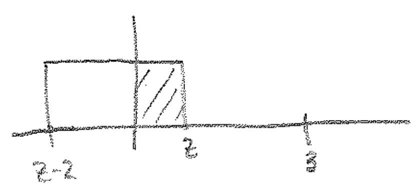
$$f_Y(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{2} & \text{if } 0 \le t \le 2\,. \\ 0 & \text{if } t > 2 \end{cases}$$

(a) Find the density function for the random variable $X + Y$ (assuming the X and Y are independent).

(b) Use this to compute the probability that $\frac{1}{2} \le X + Y \le \frac{3}{2}$.

(a) $$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(t)\, f_Y(z-t)\, dt$$

$$= \frac{1}{3}\int_0^3 f_Y(z-t)\, dt$$



$$f_Y(z-t) = \begin{cases} 0 & z-t<0 \rightsquigarrow z<t \\ 1/2 & 0 \le z-t \le 2 \rightsquigarrow z \ge t \ge z-2 \\ 0 & z-t>2 \rightsquigarrow z-2>t \end{cases}$$

$$\int_0^3 \frac{1}{3} f_{X+Y}(z-t)\,dt = \begin{cases} 0 & \text{IF } z<0 \\ \int_0^z \frac{1}{3}\cdot\frac{1}{2}\,dx & \text{IF } 0 \le z \le 2 \\ \int_{z-2}^{z} \frac{1}{3}\cdot\frac{1}{2}\,dx & \text{IF } 2<z\le 3 \\ \int_{z-2}^{3} \frac{1}{3}\cdot\frac{1}{2}\,dx & \text{IF } 3<z\le 5 \\ 0 & \text{IF } z>5. \end{cases} = \begin{cases} 0 & \text{IF } z<0 \\ z/6 & \text{IF } 0\le z\le 2 \\ 1/3 & \text{IF } 2\le z\le 3 \\ \frac{5-z}{6} & \text{IF } 3<z\le 5 \\ 0 & \text{o.w.} \end{cases}$$

(b) $$\int_{1/2}^{3/2} f_{X+Y}(t)\,dt = \int_{1/2}^{3/2} t/6\,dt = \frac{1}{6}.$$

7. (10 points.) Suppose we have an independent trials process with probability of success equal to p. Let X be the number of trials until the first success. Derive the formula for $V(X)$.

WE NEED TO DERIVE THE FORMULA FOR THE VARIANCE OF A GEOMETRIC R.V. SET $q = 1-p$. THEN:

$$E(X^2) = \sum_{k=0}^{\infty} k^2 \cdot p \cdot q^{k-1}$$

$$= \sum_{k=1}^{\infty} k^2 p \cdot q^{k-1}$$

$$= p \sum_{k=1}^{\infty} \frac{d}{dq}\left[k q^k\right]$$

$$= p \frac{d}{dq} \sum_{k=1}^{\infty} k q^k$$

$$= p \frac{d}{dq}\left(\frac{q}{1-q} \sum_{k=1}^{\infty} k\, q^{k-1} p\right)$$

$$= p \frac{d}{dq}\left(\frac{q}{1-q} \cdot \frac{1}{1-q}\right)$$

$$= p\left[\frac{1}{p^2} + \frac{2(1-p)}{p^3}\right]$$

$$= \frac{2}{p^2} - \frac{1}{p}$$

SO $V(X) = E(X^2) - \frac{1}{p^2} = \frac{1-p}{p^2}$.

8. (10 points.) We define the *moment-generating function* for a random variable X, denoted by $M_X(t)$, as:

$$M_X(t) = E\left(e^{tX}\right).$$

(a) Find the moment-generating function for X when X is a binomial random variable with n trials and probability for success p and failure q. Your answer should not include *sums* or $\cdots$.

(b) Use your answer from part (a) to find $E(X^3)$. [Hint: How is $\frac{d}{dt}M_X(t)$ related to $E(X)$?]

(a) $$\sum_{k=0}^{n} \binom{n}{k} e^{tk} p^k q^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (e^t p)^k q^{n-k}$$

$$= \left(e^t p + q\right)^n$$

BY BINOMIAL THM

(b) FROM THE HINT WE SEE $\frac{d}{dt} M_X(0) = E(X)$ AND

SIMILARLY $\frac{d^k}{dt^k} M_X(0) = E(X^k)$. SO:

$$\frac{d^3}{dt^3} M_X(0) = E(X^3)$$

START TAKING DERIVATIVES:

$$\frac{d}{dt} M_X(t) = npe^t\left(e^t p + q\right)^{n-1}$$

$$\frac{d^2}{dt^2} M_X(t) = n(n-1)(pe^t)^2\left(e^t p + q\right)^{n-2} + npe^t\left(pe^t + (1-p)\right)^{n-1}$$

$$\frac{d^3}{dt^3} M_X(t) = n \cdot n-1 \cdot n-2 (pe^t)^3 \left(e^t p + q\right)^{n-3} + 3e^{2t}(n-1)np^2\left(e^t p + q\right)^{n-2}$$

$$+ npe^t\left(e^t p + q\right)^{n-1}$$

SETTING $t=0$ GIVES: $(n-2)(n-1)\,np^3(p+q)^{n-3} + 3(n-1)np^2(p+q)^{n-2} + np(p+q)^{n-1}$