NAME: ~ ANSWER KEY ~

Math 20

Summer 2015

Exam II

Instructions:

1. Write your name *legibly* on this page.
2. There are eight problems, some of which have multiple parts. Do all of them.
3. Explain what you are doing, and show your work. You will be *graded on your work*, not just on your answer. Make it clear and legible so I can follow it.
4. It is okay to leave your answers unsimplified. That is, if your answer is the sum or product of 5 numbers, you do not need to add or multiply them. Answers left in terms of binomial coefficients or factorials are also acceptable. However, do not leave any infinite sums or products, or sums or products of a variable number of terms.
5. There are a few pages of scratch paper at the end of the exam. I *will not look* at these pages unless you write on a problem "Continued on page..."
6. This exam is closed book. You may not use notes, calculators, or any other external resource. It is a violation of the honor code to give or receive help on this exam.

1. (10 points.) Suppose that X is a continuous random variable with density function:

$$f_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{3} & \text{if } 0 \leq t \leq 3\,, \\ 0 & \text{if } t > 3 \end{cases}$$

and Y is a continuous random variable with density function:

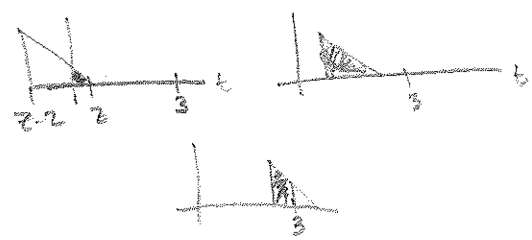
$$f_Y(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{2}t & \text{if } 0 \leq t \leq 2\,. \\ 0 & \text{if } t > 2 \end{cases}$$

Find the density function for the random variable $X + Y$ (assuming X and Y are independent).

WE HAVE: $f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(t)\, f_Y(z-t)\,dt$

$$= \frac{1}{3}\int_0^3 f_Y(z-t)\,dt$$

$$f_Y(z-t) = \begin{cases} 0 & \text{IF } z < t \\ \frac{1}{2}(z-t) & \text{IF } 0 \leq z-t \leq 2 \leadsto z \geq t \geq z-2 \\ 0 & \text{IF } z-2 > t \end{cases}$$



SO:

$$\frac{1}{3}\int_0^3 f_Y(z-t)\,dt = \begin{cases} 0 & \text{IF } z < 0 \\ \frac{1}{3}\int_0^z \frac{1}{2}(z-t)\,dt & \text{IF } 0 \leq z \leq 2 \\ \frac{1}{3}\int_{z-2}^z \frac{1}{2}(z-t)\,dt & \text{IF } 2 < z \leq 3 \\ \frac{1}{3}\int_{z-2}^3 \frac{1}{2}(z-t)\,dt \quad 3 & \text{IF } 3 < z \leq 5 \\ 0 & \text{IF } z > 5 \end{cases} = \begin{cases} 0 & \text{IF } z < 0 \\ \frac{z^2}{12} & \text{IF } 0 \leq z \leq 2 \\ \frac{1}{3} & \text{IF } 2 < z \leq 3 \\ \frac{1}{12}(-z^2+6z-5) & \text{IF } 3 < z \leq 5 \\ 0 & \text{IF } z > 5 \end{cases}$$

2. (10 points.) A box of Lucky Charms™ cereal contains 8 different types of marshmallows (can you name them all?). Let $X_1, ..., X_8$ be random variables such that X_i is the number of marshmallows of type i in a given bowl of cereal. We assume the X_is are mutually independent. We model X_i as a Poisson random variable with $E(X_1) = 5, E(X_2) = 6, E(X_3) = 4, E(X_4) = 5, E(X_5) = 2, E(X_6) = 7, E(X_7) = 6, E(X_8) = 5$.

   (a) What is the probability that your bowl of cereal contains between 35 and 42 marshmallows?

   (b) What is the expected total number of marshmallows? What is the variance?

(a) $E(X_1) + \dots + E(X_8) = 40$, SUM OF POISSON RVs IS POISSON

SO: $$P(35 \leq X_1 + \dots + X_8 \leq 42) = \sum_{k=35}^{42} \frac{40^k e^{-40}}{k!}$$

(b) $E(X_1 + \dots + X_8) = 40$, VARIANCE IS $V(X_1) + \dots + V(X_8) = E(X_1) + \dots + E(X_8) = 40$.

3. (10 points.) Let $X_1, X_2, \ldots$ be a sequence of mutually independent random variables such that $X_n = 1$ with probability $(1-2^{-n})/2$, $X_n = -1$ with probability $(1-2^{-n})/2$, $X_n = 2^n$ with probability 2^{-2n-1}, and $X_n = -2^n$ with probability 2^{-2n-1}. Show that the weak law of large numbers applies to this sequence of random variables.

WE NEED TO SHOW $\lim_{n\to\infty} P\left(\left|\frac{X_1+\cdots+X_n}{n} - E\left(\frac{X_1+\cdots+X_n}{n}\right)\right| > \varepsilon\right) = 0.$

WE NEED TO COMPUTE EXPECTATION + VARIANCE:

$$E(X_n) = \frac{(1-2^{-n})}{2} - \frac{(1-2^{-n})}{2} + \frac{2^n}{2^{2n+1}} - \frac{2^n}{2^{2n+1}}$$

$$= 0$$

$$V(X_n) = 2 \cdot \frac{(1-2^{-n})}{2} + 2 \cdot \frac{2^{2n}}{2^{2n+1}} = 2 - 2^{-n}$$

NOW $V\left(\frac{X_1+\cdots+X_n}{n}\right) = \frac{1}{n^2} \cdot \left[2n - \sum_{k=1}^{n} 2^{-k}\right]$

$$= \frac{2}{n} - \frac{1}{n^2} \sum_{k=1}^{n} 2^{-k}$$

$$\leq \frac{2}{n} - \frac{1}{n^2} \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{2}{n} - \frac{1}{n^2}$$

SO BY CHEBYSHEV'S INEQUALITY:

$$0 \leq P\left(\left|\frac{X_1+\cdots+X_n}{n}\right| \geq \varepsilon\right) \leq \frac{\left(2 - \frac{1}{n}\right)}{n\varepsilon^2}.$$

THE RHS GOES TO ZERO SO THE PROBABILITY DOES ALSO.

4. (15 points.) For each of the following, give the **best** bound for the probability. Here X is a random variable.

(a) $X \geq 0$, and $E(X) = 1$. $P(X \geq 5) \leq$ $\frac{1}{5}$

MARKOV'S INEQUALITY

(b) $E(X) = 0$ and $V(X) = 0$. $P(X \geq 5) \leq$ 0

SINCE $P(X=0)=1$.

(c) X has a symmetric distribution. $E(X) = 0$, $V(X) = 2$. $P(X \geq 3) \leq$ $\frac{1}{9}$

BY CHEBYSHEV: $P(|X| \geq 3) \leq \frac{2}{9}$

BUT BY SYMMETRY $2P(X \geq 3) = P(|X| \geq 3)$ SO

(d) $X = e$ with probability $\frac{1}{2}$ and $X = 1$ with probability $\frac{1}{2}$. $X_1, ..., X_5$ are identical independent copies of X. $P(X_1X_2X_3X_4X_5 \geq e^5) \leq$ $(\frac{1}{2})^5$

$X_1 \dots X_5 \geq e^5$ ONLY IF $X_i = e$ EACH TIME. SO

$$P(X_1 \dots X_5 \geq e^5) = \left(\frac{1}{2}\right)^5$$

SOME PEOPLE USED CHEBYSHEV, THAT WAS OK, IT JUST DIDN'T GIVE A GOOD ENOUGH BOUND...

(e) Let $X_1, X_2, ...$ be identical independent random variables with $E(X_i) = 1$ and $V(X_i) = 2$. Set $X = \lim_{n\to\infty} \frac{X_1+...+X_n}{n}$. $P(X \geq 2) \leq$ 0

BY THE STRONG LAW OF LARGE #S $X = 1$ THUS

$$P(X \geq 2) = 0$$

5. (10 points.) Let X be a uniformly distributed random variable on $[0,1]$. Let $Y = X^2+2$.

(a) What is the density function for Y?

(b) What is $E(XY)$?

(c) Is $E\left(\frac{1}{\sqrt{Y-2}}\right)$ defined? Why or why not?

(a) WE HAVE $y = \phi(X)$ w/ $\phi(X) = X^2+2$. $\phi^{-1}(y) = X$ SO

$\phi^{-1}(y) = \sqrt{y-2}$. THE CUMULATIVE DISTRIBUTION SATISFIES:

$$F_y(z) = F_X(\phi^{-1}(z)) = \sqrt{z-2} \qquad (\text{NOTE: } F_X(z) = z)$$

DIFFERENTIATING WRT z GIVES:

$$f_y(z) = \frac{1}{2\sqrt{z-2}}.$$

(b) X AND y ARE <u>NOT</u> INDEPENDENT.

$$E(Xy) = E(X^3+2X) = \int_0^1 x^3+2x\,dx = \frac{5}{4}.$$

(c) $E\left(\frac{1}{\sqrt{y-2}}\right) = E\left(\frac{1}{X}\right)$.

$E\left(\frac{1}{X}\right)$ EXISTS PROVIDED:

$\int_0^1 \frac{1}{X}\,dx$ CONVERGES, BUT $\int_0^1 \frac{1}{X}\,dx =$

$\lim_{\varepsilon \downarrow 0} \ln(x)\Big|_\varepsilon^1$, WHICH DIVERGES.

<u>NOTE:</u> SAYING $\frac{1}{0}$ IS NOT DEFINED IS NOT SUFFICIENT SINCE:

$E\left(\frac{1}{\sqrt{X}}\right) = \int_0^1 \frac{1}{\sqrt{X}}\,dx = 2$, EVEN THOUGH $\frac{1}{\sqrt{X}}$ ISN'T DEFINED FOR $X=0$.

6. (10 points.) n balls are randomly selected from an urn containing a total of N balls (without replacement). Out of the N balls in the urn a total of m are red. What is the expected number of red balls selected?

LET X = # OF RED BALLS SELECTED. SET $X_i = \begin{cases} 1 & \text{IF THE } i^{th} \text{ RED BALL IS PICKED} \\ 0 & \text{O.W.} \end{cases}$

THEN $E(X) = E(X_1 + \dots + X_m) = E(X_1) + \dots + E(X_m)$.

NOW $E(X_i) = P(i^{th} \text{ RED BALL IS PICKED})$

$$= \frac{\binom{N-1}{n-1}}{\binom{N}{n}} = \frac{n}{N}.$$

THUS: $E(X) = \sum_{i=1}^{m} \frac{n}{N} = \boxed{\frac{mn}{N}}.$

7. (12 points.)

(a) Let X be a Poisson random variable with parameter λ, where $0 < \lambda < 1$. Find $E(X!)$.

(b) Let X be a geometric random variable with parameters p for success and q for failure, show analytically that

$$P(X = n + k | X > n) = P(X = k).$$

(c) Give a verbal argument using the interpretation of a geometric random variable as to why the equation in part (b) is true.

(a) $$E(X!) = \sum_{k=0}^{\infty} \frac{k!\lambda^k e^{-\lambda}}{k!} = \sum_{k=0}^{\infty} \lambda^k e^{-\lambda} = \frac{e^{-\lambda}}{1-\lambda}.$$

(b) $$P(X=n+k \mid X>n) = \frac{P(X=n+k)}{P(X>n)} = \frac{q^{n+k-1}p}{\sum_{\ell=n+1}^{\infty} q^{\ell-1}p} = \frac{q^{n+k-1}p}{\frac{pq^n}{1-q}}$$

$$= \frac{q^{n+k-1}p}{q^n} = q^{k-1}p = P(X=k)$$

(c) GEOMETRIC RVs MODEL THE NUMBER OF TRIALS UNTIL SUCCESS OF A BERNOULLI TRIALS PROCESS. SINCE THE EXPERIMENTS IN A BERNOULLI TRIALS PROCESS ARE INDEPENDENT, THE PROB OF FAILING $n+k-1$ TIMES GIVEN THAT WE ALREADY FAILED n TIMES IS JUST THE PROB OF FAILING $k-1$ TIMES.

8. (10 points.) Let X be a Poisson random variable with parameter λ. Compute $E(X^3)$.

$$E(X^3) = \sum_{k=1}^{\infty} \frac{k^3 e^{-\lambda} \lambda^k}{k!} = \sum_{k=1}^{\infty} \frac{k^2 e^{-\lambda} \lambda^k}{(k-1)!}$$

$$= \lambda \sum_{k=0}^{\infty} \frac{(k+1)^2 e^{-\lambda} \lambda^k}{k!}$$

$$= \lambda\, E\left((X+1)^2\right)$$

$$= \lambda\, E\left(X^2 + 2X + 1\right)$$

$$= \lambda\left[E(X^2) + 2\lambda + 1\right]$$

SINCE $V(X) = E(X^2) - \lambda^2$ AND $V(X) = \lambda$ WE HAVE

$E(X^2) = \lambda + \lambda^2$ SO:

$$= \lambda\left[\lambda^2 + 3\lambda + 1\right].$$