# Midterm Exam 1

Math 20

July 19, 2012

Name: Answer Key

**Instructions:** This exam is closed-book, with no calculators, notes, or books allowed. You may not give or receive any help on the exam, though you may ask the instructor for clarification if necessary. Be sure to show all your work wherever possible. You can leave your answer in terms of factorials, binomial coefficients, fractions, etc. unless explicitly stated otherwise.

HONOR STATEMENT: I have neither given nor received any help on this exam, and I attest that all of the answers are my own work.

SIGNATURE: ____________________

| Problem | Score | Points |
|---|---|---|
| 1 | | 14 |
| 2 | | 20 |
| 3 | | 7 |
| 4 | | 9 |
| 5 | | 7 |
| 6 | | 11 |
| 7 | | 32 |
| Total | | 100 |

1. [14 points] **True or False.** For each problem, write the word **true** if the statement is always true or **false** if the statement is not always true.

(a) true A Bernoulli trial can only ever have exactly two outcomes.

(b) false If $A$ and $B$ are independent, then $P(A \cup B) = P(A) + P(B)$.

(c) true If $A^c$ and $B^c$ are independent, then so are $A$ and $B$.

(d) false If $P(A \cap B \cap C) = P(A)P(B)P(C)$, then $A$ and $B$ are independent.

(e) true Expected value is additive. That is, for random variables $X$ and $Y$,

$$E(X + Y) = E(X) + E(Y).$$

(f) false Variance is additive. That is, for random variables $X$ and $Y$,

$$V(X + Y) = V(X) + V(Y).$$

(g) false If $P(A^c) = .2$, $P(B) = .8$, and $P(A \cap B) = .7$, then $P(A \cup B) = .3$.

**2. Short Answer.** You only need to provide an answer for these. However, if you do show work, you can receive partial credit.

(a.) [5 points] What is the probability of getting exactly $n$ heads when you flip a fair coin $2n$ times?

$$b(2n, \tfrac{1}{2}, n) = \binom{2n}{n}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^n$$

(b.) [5 points] Suppose you have an urn with 34 red balls, 50 green balls and 36 yellow balls. You select 10 balls from this urn. Find the expected number of yellow balls you end up with.

Let $X = \#$ yellow balls

$$X_i = \begin{cases} 1 & \text{if ith ball is yellow} \\ 0 & \text{o.w.} \end{cases}$$

$$E(X_i) = 1 \cdot \frac{36}{120} + 0 \cdot \frac{84}{120} = \frac{36}{120}$$

$$E(X) = E(X_1 + X_2 + \cdots + X_{10}) = 10 \cdot \frac{36}{120} = 3.$$

(c.) [5 points] Suppose five people get on an elevator at the ground floor. The building has 10 floors (not including the ground floor) and each floor is equally likely to be visited by a random elevator passenger. Find the probability that at least two of the passengers will visit the same floor.

$P(\text{at least 2 visit same floor}) = 1 - P(\text{everyone visits different floor})$

It's the Birthday Problem!

$$= 1 - \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{10^5}$$

(d.) [5 points] Suppose you are playing the following game: A dealer places five cards face down on the table in random order. One of them is the King of Hearts. If you can find the King of Hearts, you win the prize. You pick a card at random. The dealer, who knows where the King of Hearts is, flips over three of the other cards. He will only flip over cards that are not the one you picked and not the King of Hearts. So there are now only two cards left face down. The dealer now gives you a chance to switch cards. Should you switch? If you switch, what is the probability that you win the prize?

It's just the Monty Hall problem w/ 5 doors!

Yes you should switch. You will win w/ probability $\frac{4}{5}$.

(There is a $\frac{4}{5}$ chance you didn't pick the right door the 1st time.)

**3.** [7 points] Prove the following formula: $P(A\cap B\cap C) = P(A)P(B|A)P(C|A\cap B)$. Show all steps.

By definition, $P(B|A) = \dfrac{P(A\cap B)}{P(A)}$

$$P(C|A\cap B) = \frac{P(A\cap B\cap C)}{P(A\cap B)}.$$

So $P(A)P(B|A)P(C|A\cap B) = \cancel{P(A)} \cdot \dfrac{\cancel{P(A\cap B)}}{\cancel{P(A)}} \cdot \dfrac{P(A\cap B\cap C)}{\cancel{P(A\cap B)}}$

$$= P(A\cap B\cap C). \quad \square$$

**4.** [9 points] Prove that if $X$ and $Y$ are independent random variables, then $V(X-Y) = V(X) + V(Y)$. Show all steps.

$$V(X-Y) = E((X-Y)^2) - E(X-Y)^2$$

$$= E(X^2 - 2XY + Y^2) - (E(X) - E(Y))^2$$

$$= E(X^2) - 2E(XY) + E(Y^2) - (E(X)^2 - 2E(X)E(Y) + E(Y^2))$$

$$= \underbrace{E(X^2) - E(X)^2}_{V(X)} + \underbrace{E(Y^2) - E(Y)^2}_{V(Y)} - 2E(XY) + 2E(X)E(Y)$$

($X,Y$ independent.)

$$V(X) + V(Y) \underbrace{- 2E(X)E(Y) + 2E(X)E(Y)}_{\text{cancel}}$$

$$= V(X) + V(Y). \quad \square.$$

5. [7 points] There are two surgeons: Dr. House and Dr. Zoidberg. Each of them perform two types of surgeries: brain surgery and removal of stitches. The following chart shows how successful each procedure has been for each doctor.

| | Dr. House | Dr. Zoidberg |
|---|---|---|
| Brain Surgery | 75% | 20% |
| Stitch Removal | 100% | 90% |
| Overall | 80% | 85% |

Notice that even though Dr. House is the better doctor when you look at the average results of each procedure, Dr. Zoidberg has the better average record overall. What is this phenomenon called? Describe the phenomenon and explain why it occurred in the case of Dr. House and Dr. Zoidberg. Your answer should not be more than a few sentences.

Simpson's Paradox. – Happens when a confounding element causes inequality to switch.

$P(\text{success} \mid \text{Dr House} \,\&\, \text{Brain Surgery}) > P(\text{Suc.} \mid \text{Dr. Z} \,\&\, \text{Brain})$

$P(\text{suc.} \mid \text{Dr. H} \,\&\, \text{Stch Rem.}) > P(\text{Suc.} \mid \text{Dr. Z} \,\&\, \text{Stch})$

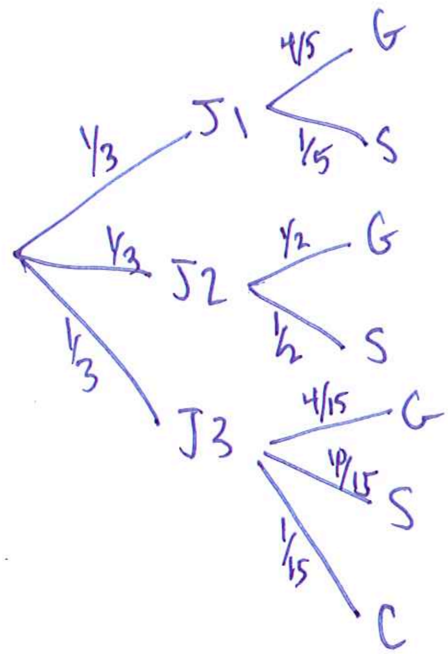
but

$P(\text{suc.} \mid \text{Dr H}) < P(\text{suc.} \mid \text{Dr. Z})$.

Happened in this case because Dr. House performs many more brain surgries than stitch removals pulling his overall average down. Dr. Zoidberg performs more stitch removals than brain surgeries pulling his overall average up.

**6.** Suppose there are 3 jars. In Jar 1, there are 4 gold coins and 1 silver coin. In Jar 2, there are 4 gold coins and 4 silver coin. In the Jar 3, there are 4 gold coins, 10 silver coins, and a copper coin.

(a.) [8 points] You pick a jar at random and draw a gold coin. What is the probability that the jar you chose was Jar 3?




So $P(J3|G) = \dfrac{P(G|J3)\cdot P(J3)}{P(G|J1)P(J1) + P(G|J2)P(J2) + P(G|J3)P(J3)}$

$$= \frac{\frac{4}{15}\cdot\frac{1}{3}}{\frac{4}{5}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3} + \frac{4}{15}\cdot\frac{1}{3}}$$

(b.) [3 points] You pick a jar at random again and draw a copper coin. What is the probability that the jar you chose was Jar 3?

$P(J3|C) = 1$, only jar w/ copper coin!

**7.** Recall that a poker hand is a set of five cards randomly drawn from a deck of 52 cards. For each of these problems, **you should justify your answer.**

(a.) [5 points] Find the probability that a poker hand does not include any face cards (so it does not contain Jack, Queen, or King).

$$\frac{\binom{40}{5}}{\binom{52}{5}}$$

\# ways to choose 5 cards from non-face cards

\# ways to choose poker hand

(b.) [7 points] Find the probability that a poker hand contains exactly two kings.

$$\frac{\binom{4}{2}\cdot\binom{48}{3}}{\binom{52}{5}}$$

pick suits of kings

pick remaining three cards from non-king cards.

\# ways to choose poker hand

(c.) [8 points] Given that the hand contains exactly two kings, find the probability that the hand doesn't contain any other pairs. (For example, if your other three cards are 4, 5, and Jack, your hand doesn't contain any other pairs. However if your other three cards are 6, 6, 9, or 4, 4, 4, then your hand does contain another pair.)

choose other 3 ranks of cards so we get no pairs

$$\frac{\binom{12}{3} \cdot 4^3}{\binom{48}{3}}$$

choose the suits for these cards.

# ways to choose other 3 cards (from non-king cards)

(d.) [12 points] Let $Y$ be the random variable equal to the number of kings in a poker hand. Find $E(Y)$ and $V(Y)$. Show all steps! You don't have to completely simplify your answer for variance, but there shouldn't be any summation signs in your answer.

$$Y = Y_1 + Y_2 + Y_3 + Y_4 + Y_5$$

where $Y_i = \begin{cases} 1 & \text{if } i^{th} \text{ card is King} \\ 0 & \text{otherwise} \end{cases}$

$$E(Y) = E(Y_1) + \cdots + E(Y_5), \quad E(Y_i) = 1 \cdot P(Y_i = 1) + 0 \cdot P(Y_i = 0) = \frac{4}{52} = \frac{1}{13}$$

$$= \frac{1}{13} + \cdots + \frac{1}{13} = \boxed{\frac{5}{13}}$$

$$V(Y) = V(Y_1 + \cdots + Y_5) = E((Y_1 + \cdots + Y_n)^2) - \underbrace{E(Y)^2}_{\left(\frac{5}{13}\right)^2}$$

$$E((Y_1 + \cdots + Y_5)^2) = E\left(\sum_{i=1}^{5} Y_i^2 + \sum_{i=1}^{5}\sum_{j \neq i} Y_i Y_j\right) = \sum_{i=1}^{5} E(Y_i^2) + \sum_{i=1}^{5}\sum_{j \neq i} E(Y_i Y_j)$$

$$= \sum_{i=1}^{5} \frac{1}{13} + \sum_{i=1}^{5}\sum_{j \neq i} \frac{1}{13} \cdot \frac{3}{51}$$

$$= \frac{5}{13} + 20 \cdot \frac{1}{13}\frac{3}{51}$$

$E(Y_i^2) = 1^2 \cdot P(Y_i = 1) + 0^2 P(Y_i = 0) = \frac{1}{13}$

$E(Y_i Y_j) = (1 \cdot 1) P(Y_i = 1, Y_j = 1) + (1 \cdot 0) P(Y_i = 1, Y_j = 0) + (0 \cdot 1) P(Y_i = 0, Y_j = 1) + (0 \cdot 0) P(Y_i = 0, Y_j = 0) = P(Y_i = 1, Y_j = 1) = \frac{4}{52} \cdot \frac{3}{51}$

Therefore,

$$\boxed{V(Y) = \frac{5}{13} + 20 \cdot \frac{1}{13} \cdot \frac{3}{51} - \left(\frac{5}{13}\right)^2}$$

**Bonus:** Pick a letter from the word MISSISSIPPI. Call this letter your "point". Replace the letter. Keep drawing letters with replacement until you draw your "point" letter again. What is the expected number of draws you will make (including the first one)?

$$\underbrace{1}_{\text{initial draw}} + \underbrace{\frac{1}{11}\cdot\frac{11}{1}}_{\text{point}=M} + \underbrace{\frac{4}{11}\cdot\frac{11}{4}}_{\text{point}=S} + \underbrace{\frac{4}{11}\cdot\frac{11}{4}}_{\text{point}=I} + \underbrace{\frac{2}{11}\cdot\frac{11}{2}}_{\text{point}=P.} = 5$$