Practice Midterm Exam

Math 20

October 2014

Name: ____

Please read the following instructions before starting the exam:

- This exam is closed book. You may not give or receive any help during the exam, though you may ask the instructors for clarification if necessary.
- Be sure to show all work whenever possible. Even if your final answer is incorrect, an appropriate amount of partial credit can be assigned.
- Please circle or otherwise indicate your final answer, if possible.
- The test has a total of 8 questions, worth a total of 100 points. Point values are indicated for each question.
- You will have two hours from the start of the exam to complete it.
- Good luck!

HONOR STATEMENT: I have neither given nor received help on this exam, and I attest that all the answers are my own work.

SIGNATURE:______________________

DOES NOT APPLY TO PRACTICE EXAMS

This page is for grading purposes only

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	
Total	

1. **Multiple choice.** Circle the correct answer to each question. [5 points each, no partial credit].

Let random variables X, Y each be picked from [0,1] with uniform density. Which of the following events is independent of $x < y$?

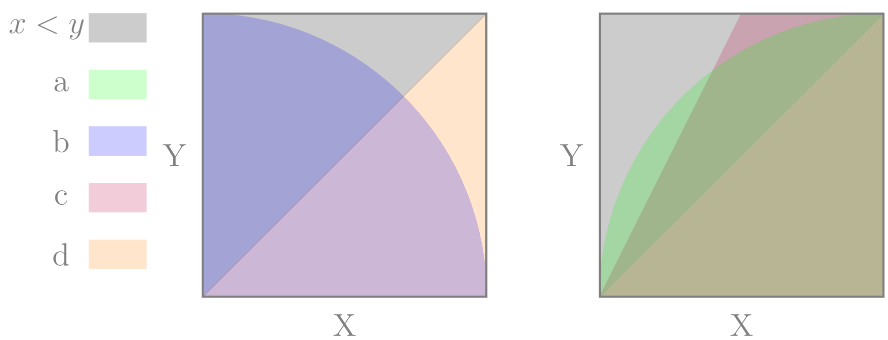
(a) $(x-1)^2 + y^2 \leq 1$

(b) $x^2 + y^2 \leq 1$

(c) $y < 2x$

(d) $y < x$

The best line of attack for a question like that is to draw the unit square:



Which is the correct Stirling's Formula?

(a) $n! \sim (ne)^{-n}\sqrt{2\pi n}$

(b) $n! \sim n^n e^{-n}\sqrt{2\pi n}$

(c) $n! \sim (ne)^{-n}\sqrt{4\pi n}$

(d) $n! \sim n^n e^{-n}\sqrt{4\pi n}$

You are about to play a racing game against your friend, and you agree to play 6 games. He's been practicing a lot more, so in any single game, the probability that you win is 0.45, and the probability that your friend wins is 0.55. What is the chance you win at least 4 times?

(a) $0.45^4 0.55^2 + 0.45^5 0.55 + 0.45^6$

(b) $\binom{6}{4} 0.45^4 0.55^2$

(c) $\binom{6}{4} 0.45^4 0.55^2 + \binom{6}{5} 0.45^5 0.55 + 0.45^6$

(d) 0.45

$$P(\geq 4) = P(4) + P(5) + P(6) = \binom{6}{4} 0.45^4 0.55^2 + \binom{6}{5} 0.45^5 0.55 + 0.45^6$$

Pick a point on a circular target with unit radius in the following way: pick a number r from $(0, 1)$ uniformly at random, and a number θ from $(0, 2\pi)$ uniformly at random. The chosen point has polar coordinates (r, θ). What is the probability that this point is within distance 0.25 from the edge of the target?

(a) $1/\sqrt{2}$

(b) $1/2$

(c) $1/4$

(d) $1/2\sqrt{2}$

It doesn't matter what the angle θ is, we are within 0.25 from the edge of the atrget if an only if $0.75 < r < 1$. Since r is picked uniformly at random from (0,1), the probability of that happening is 1/4.

2. Let the random variable X be the number of rolls of a six-sided die until, and including, the first time 6 comes up.

   - What is the probability distribution of X? [5 points]
   - What is $\mathbb{E}[X]$? [5 points]

   For any $i \leq 1$, $m(i)$ is:

   $$m(i) = \frac{1}{6}\left(\frac{5}{6}\right)^{i-1}$$

   since this means that 6 doesn not come up in any of the rolls $1, \ldots, i-1$ and it does come up on the ith roll.

   $E(X) = p \times 1 + (1-p)(1+E(X)) = 1 + (1-p)E(X)$

   $(1 - 1 + p)E(X) = 1$

   $E(X) = 1/p = 6$

3. Find the cumulative probability distributions $F(x) = P[X \leq x]$ for each of the following probability densities: [15 points]

   - $f_1(x) = \lambda e^{-\lambda x}$, $x \in (0, \infty)$
   - $f_2(x) = 1/a,\ x \in (0, a)$
   - $f_3(x) = 1/x^2,\ x \in (1, \infty)$

$$P_[X \leq x] = \int_0^x \lambda e^{-\lambda t} dt = \lambda \left[\frac{e^{-\lambda t}}{-\lambda} \right]_0^x = \left[-e^{-\lambda t} \right] = 1 - e^{-\lambda x}$$

$$P_2[X \leq x] = \int_0^x \frac{1}{a} dt = \frac{t}{a}$$

$$P_3[X \leq x] = \int_1^x \frac{1}{t^2} dt = \left[\frac{-1}{t} \right]_1^x = 1 - \frac{1}{x}$$

4. You throw $6n$ six-sided dice. What is the probability that each of $\{1,2,3,4,5,6\}$ appears exactly n times? [10 points]

Consider all the possible results of throwing $6n$ six-sided dice. There are 6^{6n} of them. How many are there such that there are exactly n of each number? There are $\binom{6n}{n}$ ways to pick n dice to have 1 on them. Then there are $\binom{5n}{n}$ ways to pick another n dice to have 2 on them. Then there are $\binom{4n}{n}$ ways to pick another n dice to have 3 on them and so on. So the answer is:

$$\frac{\binom{6n}{n}\binom{5n}{n}\binom{4n}{n}\binom{3n}{n}\binom{2n}{n}\binom{n}{n}}{6^{6n}}$$

5. Pick a permutation of $\{1, 2, 3, 4, 5, 6\}$ uniformly at random. What is the probability that it has a fixed point that is an even number? [15 points]

The probability that 2 is a fixed point is 1/6, since there are 5! permutations that have 2 in second spot, and 6! permutations in total. Similarly, the probability that 4 is a fixed point is 1/6, and so is the probability that 6 is a fixed point.

The probability that both 2 and 4 are fixed points is 4!/6!=1/30, and so is the probability for any two other specific numbers. The probability that all 2, 4, and 6 are fixed is 3!/6!=1/120. So by inclusion exclusion, the answer is:

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} - \frac{1}{30} - \frac{1}{30} - \frac{1}{30} + \frac{1}{120} = \frac{49}{120}.$$

6. Recall that the Poisson distribution is:

$$P(X = k, \lambda) = \lim_{n\to\infty} \binom{n}{k} (\lambda/n)^k (1 - \lambda/n)^{n-k}.$$

Use the fact that $P(X = 0, \lambda) = e^{-\lambda}$ to derive an expression for $P(X = k, \lambda)$ that is independent of n. [10 points]

For any $k > 0$:

$$\frac{P(X = k, \lambda)}{P(X = k-1, \lambda)} = \lim_{n\to\infty} \frac{\binom{n}{k}(\lambda/n)^k(1-\lambda/n)^{n-k}}{\binom{n}{k-1}(\lambda/n)^{k-1}(1-\lambda/n)^{n-k+1}} = \lim_{n\to\infty} \frac{\frac{n!}{k!(n-k)!}}{\frac{n!}{(k-1)!(n-k+1)!}} \frac{\lambda/n}{1-\lambda/n}$$

$$= \lim_{n\to\infty} \frac{n-k+1}{k} \frac{\lambda/n}{1-\lambda/n} = \lim_{n\to\infty} \frac{\lambda - (k-1)\lambda/n}{k(1-\lambda/n)}$$

As $n \to \infty$ all terms with $1/n$ go to 0. So:

$$\frac{P(X = k, \lambda)}{P(X = k-1), \lambda} = \frac{\lambda}{k}$$

If we combine this with the fact that $P(X = 0, \lambda) = e^{-\lambda}$, we get:

$$P(X = 0, \lambda) = e^{-\lambda}$$

$$P(X = 1, \lambda) = e^{-\lambda}\lambda$$

$$P(X = 2, \lambda) = e^{-\lambda}\frac{\lambda^2}{2}$$

$$P(X = 3, \lambda) = e^{-\lambda}\frac{\lambda^3}{6}$$

$$\vdots$$

$$P(X = k, \lambda) = e^{-\lambda}\frac{\lambda^k}{k!}$$

7. Let X and Y be independent random variables with density functions:

- $f_X(x) = 3/x^4,\ \Omega_X = (1, \infty)$
- $f_Y(y) = e^{-y},\ \Omega_Y = (0, \infty)$

Find the expectation and variance of $Z = X + Y$. [10 points]

$$E(X) = \int_1^\infty \frac{3x}{x^4} dx = \int_1^\infty \frac{3}{x^3} dx = \left[\frac{-3}{2x^2}\right]_1^\infty = \frac{3}{2} = \mu_X$$

$$V(X) = E(X^2) - \mu_X^2 = \int_1^\infty \frac{3x^2}{x^4} dx - \frac{9}{4} = \int_1^\infty \frac{3}{x^2} dx - \frac{9}{4} = \left[\frac{-3}{x}\right]_1^\infty - \frac{9}{4} = 3 - \frac{9}{4} = \frac{3}{4}$$

$$E(Y) = \int_0^\infty y e^{-y} dy = \left[-e^{-y} - y e^{-y}\right]_0^\infty = e^0 = 1 = \mu_Y$$

$$V(Y) = E(Y^2) - \mu_Y^2 = \int_0^\infty y^2 e^{-y} dy - 1 = \left[-y^2 e^{-y} - 2e^{-y} - 2y e^{-y}\right]_0^\infty - 1 = 2e^0 - 1 = 2 - 1 = 1$$

$$E(Z) = E(X) + E(Y) = \frac{5}{2}$$

$$V(Z) = V(X) + V(Y) = \frac{5}{3}$$

8. Let X be a continuous random variable that is uniformly distributed on $(0, 2)$. Find the sample space, probability density and cumulative probability distribution of

$$Y = X^3.$$

[10 points]

$\Omega_Y = (0, 8)$. To find the cumulative distribution function of Y, we use the fact that X^3 is an increasing function.

$$P[Y \leq y] = P[X \leq \sqrt[3]{y}] = \frac{1}{2}\sqrt[3]{y}$$

And the probability density $f_Y(y)$ is the derivative of cumulative probability distribution:

$$f_Y(y) = \frac{d}{dy}\left(\frac{1}{2}\sqrt[3]{y}\right) = \frac{1}{6}y^{-2/3}$$