Math 20 Fall 2013

Probability

Midterm Exam

Wednesday November 6th 7:00-9:00 PM

Your name (please print): ____

Instructions: This is a closed book exam. You may not refer to the textbook or any notes, however you may use a calculator. Answers do not need to be simplified, however you should **provide enough work to explain how you arrived at your answer.**

The Honor Principle requires that you neither give nor receive any aid on this exam.

For grader use only:

Problem	Points	Score
1	12	
2	6	
3	6	
4	8	
5	10	
6	9	
Total	51	

1. a. You arrive at KAF to get coffee and a chocolate chip cookie and find that there is one person currently in line in addition to the person currently ordering. You know from experience that the amount of time it takes one person to order at KAF is exponentially distributed and averages one minute. What is the probabilty that you have to wait longer than 3 minutes to order?

The time for each each person to order is given by an exponential distribution with parameter $\lambda = 1$. Note that even though the first person is already ordering, the memoryless property of the exponential distribution means that the time we have to wait for that person is still given by the same distribution. So we need to look at the convolution of an exponential distributions, $F_Y(x)$ both with $\lambda = 1$.

$$F_X(x) = \int_{-\infty}^{\infty} F_Y(y)F_Y(x-y)dy = \int_0^x e^{-y}e^{y-x}dy = \int_0^x e^{-x}dy = xe^{-x}$$

Hence, the probability we must wait longer than 3 minutes is

$$P(X > 3) = \int_3^{\infty} xe^{-x}dx = (-1-x)e^{-x}\Big|_3^{\infty} \approx 0.1991$$

b. The Advance Transit bus to Lebanon comes by every 15 minutes. (Assume this is exact.) Assuming you show up at the bus stop at a random time, let T be a random variable for the amount of time that you must wait for the bus. Write down the probability density function for T.

Any value between 0 and 15 is equally likely, so

$$F_T(x) = \begin{cases} \frac{1}{15} & 0 \le x \le 15 \\ 0 & \text{otherwise} \end{cases}$$

c. As soon as you get your order at KAF, you want to catch the bus down to Lebanon. Find the probability density function for the total amount of time you must spend waiting, both at KAF and for the bus to Lebanon.

We need to find the sum $Z = X + T$ of the time it takes to order, with the time we must wait for the bus. So the distribution function for Z is given by the convolution of the distribution functions for these two variables,

$$F_Z(x) = F_X * F_T(x) = \int_{-\infty}^{\infty} F_X(x-y)F_T(y)dy$$

Now, $F_X(x-y)$ is nonzero so long as $0 \le x - y$, (or $x \ge y$) and $F_T(y)$ is nonzero so long as $0 \le y \le 15$. So

$$F_Z(x) = \begin{cases} \int_0^x \frac{1}{15} \cdot (x-y)e^{y-x)}dy & 0 \le x \le 15 \\ \int_0^{15} \frac{1}{15} \cdot (x-y)e^{y-x}dy & 15 \le x \end{cases}$$

or

$$F_Z(x) = \begin{cases} \frac{1}{15}(1 - xe^{-x} - e^{-x}) & 0 \le x \le 15 \\ \frac{1}{15}\left(e^{15-x}(x-14) - e^{-x}(x+1)\right) & 15 \le x \end{cases}$$

2. Once you get on the bus, you eat the chocolate chip cookie (that you bought at KAF in problem 1) and find that it contained only 6 chocolate chips in it. What is the probability of this happening if KAF chocolate chip cookies average 10 chocolate chips? (Assume that chocolate chips end up in each cookie randomly, independent of one another.)

   Use the Poisson distribution with $\lambda = 10$.

$$P(X = 6) = \frac{10^6}{6!} e^{-10} \approx 0.0631$$

3. Suppose you eat 50 KAF chocolate chip cookies in a quarter. Use the central limit theorem to estimate the probability that you have consumed between 510 and 520 (inclusive) chips total.

   Let X_i be the number of chips in cookie i, and $S_{50} = X_1 + X_2 + \cdots + X_{50}$ the total number of chips. Then S_{50} is Poisson distributed with $\lambda = 500 = \mathbb{E}(S_{50}) = Var(S_{50})$. By the Central Limit Theorem:

$$\begin{aligned} P(510 \leq S_50 \leq 520) &= P\left(\frac{509.5 - 500}{\sqrt{500}} \leq S_50^* \leq \frac{520.5 - 500}{\sqrt{500}} \right) \\ &= P(0.42 < S_50^* < 0.92) \\ &\approx 0.3212 - 0.1628 = 0.1584 \end{aligned}$$

4. Let $\{X_i\}$ be a sequence of random variables each with distribution function

$$P(X = k) = \begin{cases} 0.6 - 0.1 \cdot k & 2 \leq k \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

Let $S_n = X_1 + X_2 + \cdots X_n$.

Use the central limit theorem to estimate

a. $P(50 < S_{20} < 70)$

Compute: $\mathbb{E}(X_i) = 2 \cdot 0.4 + 3 \cdot 0.3 + 4 \cdot 0.2 + 5 \cdot 0.1 = 3.$
$Var(X_i) = \mathbb{E}(X_i^2) - \mathbb{E}(X_i)^2.$
$\mathbb{E}(X_i^2) = 4 \cdot 0.4 + 9 \cdot 0.3 + 16 \cdot 0.2 + 25 \cdot 0.1 = 10.$
So $Var(X_i) = 10 - 9 = 1.$
Using the Central Limit Theorem,

$$\begin{aligned} P(50 < S_{20} < 70) &= P\left(\frac{50.5 - 60}{\sqrt{20}} < S_{20}^* < \frac{69.5 - 60}{\sqrt{20}}\right) \\ &\approx P(-2.12 < S_{20}^* < 2.12) = 2 * 0.4830 = 0.966 \end{aligned}$$

b. $P(S_{100} < 320)$

Using the Central Limit Theorem,

$$\begin{aligned} P(S_{100} < 320) &= P\left(S_{100}^* < \frac{319.5 - 300}{\sqrt{100}}\right) \\ &\approx P(S_{100}^* < 1.95) = 0.5 + 0.4744 = 0.9744 \end{aligned}$$

5. For each part below, give the best upper bound you can for the probability $P(X \geq 5)$ that the random variable X is greater than or equal to 5.

   a. $X > 0$ and $\mathbb{E}(X) = 3$. We can use Markov's inequality:

$$P(X \geq 5) \leq \frac{3}{5} = 0.6$$

   b. X has a symmetric density function and $\mathbb{E}(X) = 0$. Since X is symmetric, we know that $P(X < 0) = 0.5$, so $P(X \geq 5) < 0.5$, however we could have, for example, $X = \pm 10$, so this is best possible.

   c. $\mathbb{E}(X) = 2$ and $Var(X) = 2$. We can use Chebyshev's inequality:

$$P(X \geq 5) \leq P(|X - 2| \geq 3) \leq \frac{Var(X)}{3^2} = \frac{2}{9}$$

   d. $X = \lim_{n\to\infty} \frac{X_1+X_2+\ldots X_n}{n}$ where $\mathbb{E}(X_i) = 4$ and $Var(X_i) = 2$ for all i. By the weak law of large numbers:

$$P(|\lim_{n\to\infty} \frac{S_n}{n} - 4| < 0.5) = 1$$

   So

$$P(X \geq 5) = 0$$

   e. $\mathbb{E}(X) = 3$. This really tells us nothing, we can construct distributions with expected value 3, but arbitrarily large probability of being greater than 5, so the only bound we can give is

$$P(X \geq 5) < 1$$

6. Let X be a binomially distributed random variable with parameters n and p.

   a. What is $\mathbb{E}(X^2)$? We know $\mathbb{E}(X) = np$ and $Var(X) = np(1-p)$. We also know $Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$. Solving for $\mathbb{E}(X^2)$,

$$\mathbb{E}(X^2) = \mathbb{E}(X)^2 + Var(X) = n^2p^2 + np(1-p) = np(1-p+np)$$

   b. Show that

$$\mathbb{E}\left(\frac{1}{1+X}\right) = \frac{1-(1-p)^{n+1}}{(n+1)p}$$

$$\begin{aligned}
\mathbb{E}\left(\frac{1}{1+X}\right) &= \sum_{i=0}^{n} \frac{1}{1+i}\left(\frac{n!}{i!(n-i)!}\right)p^i(1-p)^{n-i} \\
&= \sum_{i=0}^{n} \frac{n!}{(i+1)!(n-i)!}p^i(1-p)^{n-i} \\
&= \frac{1}{(n+1)p}\sum_{i=0}^{n} \frac{(n+1)!}{(i+1)!(n-i)!}p^{i+1}(1-p)^{n-i} \\
&= \frac{1}{(n+1)p}\sum_{i=0}^{n} \binom{n+1}{i+1}p^{i+1}(1-p)^{(n+1)-(i+1)} \\
&= \frac{1}{(n+1)p}\underbrace{\sum_{j=1}^{n+1} \binom{n+1}{j}p^j(1-p)^{n+1-j}}_{\substack{\text{Would be a complete sum of the binomial} \\ \text{distribution for } n+1 \text{ if it included } j=0}} \qquad \text{(change of variable)} \\
&= \frac{1}{(n+1)p}(1-(1-p)^{n+1})
\end{aligned}$$