Final exam due Wednesday afternoon, December 11 under the door of 105 Choate House or by blitzmail directly to Jim Baumgartner. Please write carefully and express yourself clearly.

1. Suppose you have two coins. One comes up heads with probability p and tails with probability q; the other has probabilities $\bar{p}$ and $\bar{q}$. If each coin is labeled 1 on the heads side and 2 on the tails side, then tossing both coins results in a total between 2 and 4. Let X be the random variable giving this total.
   (a) In terms of p, q, $\bar{p}$ and $\bar{q}$, find $P(X = 2)$, $P(X = 3)$, and $P(X = 4)$.
   (b) Show that the coin can be loaded (i.e., p and $\bar{p}$ can be adjusted) so that $P(X = 2) = P(X = 4) = a$ for some $a \neq \frac{1}{4}$.
   (c) Is it possible to adjust p and $\bar{p}$ so that $P(X = 2) = P(X = 3) = P(X = 4)$?

2. A certain coin has either probability $p = \frac{1}{2}$ or $p = \frac{1}{3}$ of coming up heads, but we don't know which. Thus we think of tossing the coin n times and evaluating the total S_n of times it comes up heads to see if that number is either more or less than a target value k. Find the smallest possible value of n so that for some k, if $p = \frac{1}{2}$ then $P(S_n > k) > .95$, and if $p = \frac{1}{3}$ then $P(S_n < k) > .95$.

3. One day, while walking in the grass at Hogwarts, Harry Potter discovers a magic coin. He calls it a heads-loving coin since it never produces two tails in a row. If it comes up heads, the next toss could be heads or tails with equal probability, but if it comes up tails the next toss is sure to be heads.

   Harry discovers that the heads-loving coin is also called a Markov coin by some wizards. What is the reason for this? How likely is it that the coin will come up heads in the long term (i.e. for a large number of tosses)?

4. Suppose you have two coins. One is a standard coin (which produces heads with probability $\frac{1}{2}$ on each toss) and the other is a Harry Potter heads-loving coin, but you don't know which is which. Are there numbers k and n as in exercise 2 which allow you to determine the identity of each coin with probability 95%? Explain.

5. Let us consider playing a board game in which we have to move a distance of 6 squares. This we do by repeatedly rolling a die, moving the number of squares showing each time. The rules, however, insist that we must land on the last square exactly. Thus, for example, if only one square remains, we may not move until we toss a 1. This problem may be regarded as a Markov chain with states 0, 1, 2, 3, 4, 5, and 6 giving the number of squares moved so far. Find the transition matrix. What is the time to absorption for each state? (Note that state 6 is an absorbing state.) How do you explain your answer?