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- So far we have dealt mostly with independent trials processes.
- Now, we will study *Markov chains*, a process in which the outcome of a given experiment can affect the outcome of the next experiment.

The Setup

- We have a set of **states**, $S = \{s_1, s_2, \dots, s_r\}$.
- The process starts in one of these states and moves successively from one state to another.
- Each move is called a **step**.
- We denote the random variable X_i to be the state of process at step i
- If the chain is currently in state s_i, then it moves to state s_j at the next step with a probability denoted by p_{ij}

$$p_{ij} = P(X_k = s_j | X_{k-1} = s_i)$$

The Setup

- We have a set of **states**, $S = \{s_1, s_2, \ldots, s_r\}$.
  Sometimes we just take $s_1 = 1, s_2 = 2, \ldots, s_r = r$
- X_i is the state of process at step i, X_i takes values in S.
- If the chain is currently in state s_i, then it moves to state s_j at the next step with a probability denoted by p_{ij}

$$p_{ij} = P(X_k = s_j | X_{k-1} = s_i)$$

- This probability does not depend upon which states the chain was in before the current state.

$$P(X_k = x_k | X_{k-1} = x_{k-1}, X_{k-2} = x_{k-2}, \ldots, X_0 = x_0) = P(X_k = x_k | X_{k-1} = x_{k-1})$$

  This known as the **Markov property**.

The Setup continued

- the process can remain in the same state s_i, with probability

$$p_{ii} = P(X_k = s_i, X_{k-1} = s_i)$$

- p_{ij} are called the **transition probabilities**
- We can store all the p_{ij} in a $r \times r$ matrix, known as the transition matrix, where $r = \#S$.
  Where each row adds up to 1,

$$\sum_{j=1}^{r} p_{ij} = 1$$

- To start the process, we give an initial probability distribution for starting state, a distribution for X_0.

Example: The Land of Oz weather

The Land of Oz is blessed by many things, but not by good weather.

- They never have two nice days in a row.
- If they have a nice day, they are just as likely to have snow as rain the next day.
- If they have snow or rain, they have an even chance of having the same the next day.
- If there is change from snow or rain, only half of the time is this a change to a nice day.

Step 1: identify the different states i.e. the kinds of weather.
Call these R, N, and S.
Step 2: write down probabilities of moving from one state to another
Step 3: Create a transition matrix

Example: The Land of Oz weather

Step 1: identify the different states i.e. the kinds of weather.
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Step 2: write down probabilities of moving from one state to another
Step 3: Create a transition matrix

$$P = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

- Given that today we have nice weather, what is the probability that it will snow in two days?

Multiple steps

- Given that the chain is in state s_i, what is the probability it will be in state j *two steps* from now?
  Denote this probability by $p_{ij}^{(2)}$.
- In the previous example, we saw:

$$p_{23}^{(2)} = p_{21}p_{13} + p_{22}p_{23} + p_{23}p_{33}$$

- What is the generic formula for $p_{ij}^{(2)}$?

$$\begin{aligned} p_{ij}^{(2)} &:= P(X_2 = j | X_0 = i) && \text{(definition)} \\ &= \sum_{k=1}^{r} P(X_2 = j | X_1 = k, X_0 = i) P(X_1 = k | X_0 = j) && \text{(conditioning on } X_1) \end{aligned}$$

Multiple Steps

- Given that the chain is in state i, what is the probability it will be in state j *two steps* from now?
  Denote this probability by $p_{ij}^{(2)}$.

$$
\begin{aligned}
p_{ij}^{(2)} &:= P(X_2 = s_j | X_0 = s_i) && \text{(definition)} \\
&= \sum_{k=1}^{r} P(X_1 = s_k | X_0 = s_i) P(X_2 = s_j | X_1 = s_k, X_0 = s_i) && \text{(conditioning on } X_1\text{)} \\
&= \sum_{k=1}^{r} P(X_1 = s_k | X_0 = s_i) P(X_2 = s_j | X_1 = s_k) && \text{(Markov property)} \\
&= \sum_{k=1}^{r} p_{ik} p_{kj}
\end{aligned}
$$

Multiple Steps

Theorem

- Let P be the transition matrix of a Markov chain.
- Let $p_{ij}^{(n)}$ denote the probability that the Markov chain will be in state j in n steps from now, given that now is in the state i.

The probability $p_{ij}^{(n)}$ is given by the (i, j)-entry of the matrix P^n.
In short,

$$P(X_{k+n} = s_j | X_k = s_i) := p_{ij}^{(n)} = (P^n)_{i,j}.$$

Back to the Land of Oz

- $P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0.5 & 0.25 & 0.25 \\ 0.5 & 0. & 0.5 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}$
- $P^2 = \begin{pmatrix} \frac{7}{16} & \frac{3}{16} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{16} & \frac{7}{16} \end{pmatrix} = \begin{pmatrix} 0.4375 & 0.1875 & 0.375 \\ 0.375 & 0.25 & 0.375 \\ 0.375 & 0.1875 & 0.4375 \end{pmatrix}$
- $P^3 = \begin{pmatrix} \frac{13}{32} & \frac{13}{64} & \frac{25}{64} \\ \frac{13}{32} & \frac{3}{16} & \frac{13}{32} \\ \frac{25}{64} & \frac{13}{64} & \frac{13}{32} \end{pmatrix} = \begin{pmatrix} 0.40625 & 0.203125 & 0.390625 \\ 0.40625 & 0.1875 & 0.40625 \\ 0.390625 & 0.203125 & 0.40625 \end{pmatrix}$
- $P^4 = \begin{pmatrix} \frac{103}{256} & \frac{51}{256} & \frac{51}{128} \\ \frac{51}{128} & \frac{13}{64} & \frac{51}{128} \\ \frac{51}{128} & \frac{51}{256} & \frac{103}{256} \end{pmatrix} = \begin{pmatrix} 0.402344 & 0.199219 & 0.398438 \\ 0.398438 & 0.203125 & 0.398438 \\ 0.398438 & 0.199219 & 0.402344 \end{pmatrix}$

Back to the Land of Oz

- $P^5 = \begin{pmatrix} 0.400391 & 0.200195 & 0.399414 \\ 0.400391 & 0.199219 & 0.400391 \\ 0.399414 & 0.200195 & 0.400391 \end{pmatrix}$
- $P^{10} = \begin{pmatrix} 0.400001 & 0.2 & 0.4 \\ 0.4 & 0.200001 & 0.4 \\ 0.4 & 0.2 & 0.400001 \end{pmatrix}$
- $P^{100} = \begin{pmatrix} 0.4 & 0.2 & 0.4 \\ 0.4 & 0.2 & 0.4 \\ 0.4 & 0.2 & 0.4 \end{pmatrix}$

After 100 days, no matter what was the weather on the first day, the probability of getting a nice day is only 20 percent ☹
How would you write that formally?

Example - Broken Phone

The President of the United States tells person A his or her intention to run or not to run in the next election. Then A relays the news to B, who in turn relays the message to C, and so forth, always to some new person. We assume that there is a probability a that a person will change the answer from yes to no when transmitting it to the next person and a probability b that he or she will change it from no to yes.

We choose as states the message, either yes or no.

The initial state represents the President's choice.

- Find the transition matrix.

The transition matrix is $P = \begin{array}{c c} & \begin{array}{cc} Yes & No \end{array} \\ \begin{array}{c} yes \\ no \end{array} & \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} \end{array}$

Example - Broken Phone

$$P = \begin{array}{c c} & \begin{array}{cc} Yes & No \end{array} \\ \begin{array}{c} yes \\ no \end{array} & \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} \end{array}$$

- Calculate P^n for several n and for different values of a and b.

Ehrenfest model (diffusion of gases.)

We have two urns that, between them, contain four balls. At each step, one of the four balls is chosen at random and moved from the urn that it is in into the other urn.

- How would you model this as a Markov chain?
  We choose, as states, the number of balls in the first urn.
- Find the transition matrix.

$$P = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Probability vector

- A probability vector with r components is a row vector whose entries are non-negative and sum to 1.
- We are interested in the long-term behavior of a Markov chain when it starts in a state chosen by a probability vector.
- If u is a probability vector which represents the initial state of a Markov chain, then we think of the ith component of u as representing the probability that the chain starts in state s_i.

$$u = (P(X_0 = s_1), P(X_0 = s_2), \ldots, P(X_0 = s_r))$$

- How do write the probability vector which represents the state of a Markov chain at the nth step?

Probability distribution for X_n

Theorem

Let

- P be the transition matrix of a Markov chain, and
- u be the probability vector which represents the starting distribution.

Then the probability that the chain is in state s_i after n steps is the ith entry in the vector $u \cdot P^n$. In other words,

$$(P(X_n = s_1), P(X_n = s_2), \ldots, P(X_n = s_r)) = u^{(n)} = u \cdot P^n$$

Absorbing Markov Chains

- A state s_i of a Markov chain is called *absorbing* if it is impossible to leave it (i.e., $p_{ii} = 1$).
- Markov chain is absorbing if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state (not necessarily in one step).
- In an absorbing Markov chain, a state which is not absorbing is called *transient*.