Math 20, Fall 2017

Edgar Costa

Week 3

Dartmouth College

Random Stuff

- Tutorial: Thursdays 7-8:30 pm
- Office hours: M: 5:30 - 7+ pm, Th: 8 - 9 am, and by appointment
- Study groups: Sundays 3:30pm - 5:00pm

- Midterm in one week!

The hat problem (GS: Example 3.12)

In a restaurant n hats are checked and they are hopelessly scrambled; what is the probability that no one gets his own hat back?

- What is the probability that the i-th person gets her hat back?
- What is the probability that the i-th and the j-th person get their hat back?
- What is the probability that at least someone gets their hat back?

$$\binom{n}{1}\frac{(n-1)!}{n!} - \binom{n}{2}\frac{(n-2)!}{n!} + \binom{n}{3}\frac{(n-3)!}{n!} \cdots + (-1)^{n-1}\binom{n}{n}\frac{1}{n!}$$

$$\xrightarrow[n \to +\infty]{} \frac{1}{e}$$

Monty Hall problem (M: Section 2.4.2 or GS: Example 4.6)

Problem

A game-show host offers you the choice of three doors. Behind one of these doors is the grand prize, and behind the other two are goats. The host (who knows what is behind each of the doors) announces that after you select a door (without opening it), he will open one of the other two doors and purposefully reveal a goat. You select a door. The host then opens one of the other doors and reveals the promised goat. He then offers you the chance to switch your choice to the remaining door.

To maximize the probability of winning the grand prize, should you switch or not? Or does it not matter?

3 reasonings

1. Once the host reveals a goat, the prize must be behind one of the two remaining doors. Since the prize was randomly located to begin with, there must be equal chances that the prize is behind each of the two remaining doors. The probabilities are therefore both 1/2, so it doesn't matter if you switch. (Think as someone who just entered the room)
2. There is initially a 1/3 chance that the prize is behind any of the three doors. So if you don't switch, your probability of winning is 1/3. No actions taken by the host can change the fact that if you play a large number n of these games, then (roughly) $n/3$ of them will have the prize behind the door you initially pick.
3. If you don't switch, your probability of winning is 1/3. However, if you switch, your probability of winning is greater than 1/3. It increases to 2/3.

Which reasoning is correct? (Draw tree diagrams!)

Independence and conditional

How would you define the independence of two random variables X and Y?

X and Y are independent random variables if

$$m_{(X,Y)}(x, y) = m_X(x)m_Y(y)$$

Is $X|A$ a random variable?

Upcoming

- Friday Sep 29, Prof. Jay Pantone will cover for me.
- Monday, Oct 2, 1st midterm!
  - Arrive to the classroom at least 5 mins before
  - Covers everything before Today
  - David Morin's book is full of problems with solutions
  - I have a bunch of old exams from other instructors on my website
  - T/F $\sim$ 25% + Fill in the blank $\sim$ 25% + Free answer $\sim$ 50%
- Tuesday Oct 3rd, new worksheet!
- Wednesday Oct 4, Prof. Kate Moore will cover for me.
  HW is due
- Friday Oct 6, Prof. Jay Pantone will cover for me.
  Worksheet is due

$$\Omega = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_n, \ldots\}$$

If Ω is infinite but countable, i.e., we can enumerate the events, then everything is the same. However, the sums may turn to series, and they must converge.

Example:

X = how many times I have to toss a coin to get tails

- $P(X = 1) = p$
- $P(X = 2) =(1 - p)p$
- $P(X = k) = (1 - p)^{k-1}p$
- $P(X \leq k) =1 - (1 - p)^k \to ?$
- $P(X > k + l | X > k) =P(X > l)$ (the random variable is memoryless)

Bernoulli trial process

Definition

A **Bernoulli trial** (or binomial trial) is a random experiment with exactly two possible outcomes, "success" and "failure", in which the probability of success is the same every time the experiment is conducted.

Definition

A **Bernoulli trial process** is a sequence of independent identically distributed Bernoulli trials.

We can write $X = \sum_{i=1}^{n} X_i$, where X_i are i.i.d. (independent and identically distributed) Bernoulli trials, i.e., all with the same probability of success.

This also known as the **binomial distribution**, and we write $X = \text{Binomial}(n, p)$.

Definition

X = number of trials in a sequence of iid Bernoulli trials needed to get one success

= **Geometric distribution** = Geometric(p)

$$\Omega = \{1, 2, \ldots, \}$$

$$P(X = k) = (1-p)^{k-1} p$$

We saw this example two slides ago!

Examples: Bernoulli trial process

Example 1

A die is rolled 30 times, what is the probability that 6 is observed exactly 5 times?

What is the most probable number of times that 6 will turn up?

Example 2

In a multiple choice exam with 50 questions, where each question has 2 choices, what is the probability of guessing k questions correctly?

More examples

Exercise GS 3.2.11

A restaurant offers apple and blueberry pies and stocks an equal number of each kind of pie. Each day ten customers request pie. They choose, with equal probabilities, one of the two kinds of pie. How many pieces of each kind of pie should the owner provide so that the probability is about .95 that each customer gets the pie of his or her own choice?

Negative binomial distribution

Definition

X = number of trials in a sequence of iid Bernoulli trials needed to get r success.

- $\Omega = ?$
- $P(X = k) = ?$
- $r = 1 \rightsquigarrow X = \text{Geometric}(p)$
- Why $\sum_{k \in \Omega} P(X = k) = 1$?