Math 20, Fall 2017

Edgar Costa

Week 1

Dartmouth College

Introduction

- Website: `www.math.dartmouth.edu/~m20f17/`
- Canvas only for grades and email
- We will use all the X-hours.
- Grading:
  - Homework - 100 pts
  - 3 in class midterms - 80 + 80 + 80 pts
  - Final 160 pts
- Midterms dates: Oct 2, Oct 16, Oct 30 (all Mondays)
- Homework, usually due Wednesdays and Fridays, always by the beginning of class.
  You already have homework for tomorrow!

Textbooks

- David Morin - Probability: for the enthusiastic beginner
  Very useful in the beginning!
- Charkes Grinstead, Laurie Snell - Introduction to Probability
  Free!, pdf on the website.
  It will play a bigger role after the first half.

Formalism

From OCR:

This course will serve as an introductions to the ***foundations*** *of probability theory.*

For good foundations we need have some formalism.

$\rightsquigarrow$

This requires to read and write proofs!
This will serve as a gateway to develop your critical thinking.

Tomorrow over the X-hour **you** will practice writing proofs. Be sure to do your homework!

Why do we care about probability?

Probability, is a way to turn hard problems into simpler ones.

For example, in science, we usually observe a dichotomy between structure and randomness. Given a problem, either:

- the answer is systematic, due to some rigidity of the problem, or
- the answer is complex and hard to determine, and in that case, it behaves “randomly” according to some probabilistic law.

Thus, instead of trying to understand the ”complex” solutions, we might instead focus on understanding its “random behaviour”.

Examples

We will mostly focused on studying the “random behaviour” of an experiment

For example:

- Coin toss
- Rolling a die

In theory, we could predict the outcome of each experiment. However, very small changes in the initial conditions produce totally different outcomes.

Other examples:

- Weather
- Stock market

Definition of Probability (for the first two weeks)

Definition

Consider a very large number of identical trials of a certain process; for example, flipping a coin, rolling a die, picking a ball from a box (with replacement), etc.

We say that an event E (for example, getting a Heads, rolling a 5, or picking a blue ball) has probability p if the event E occurs a p fraction of the trials, on average.

We write $P(E) = p$.

Remark: This definition is indeed makes sense by the law of large numbers, something we will learn in a couple of weeks.

Examples

- Experiment: Tossing a fair coin
  $P(\text{Heads}) = 1/2$
- Experiment: rolling a fair die
  $P(\text{rolling } 5) = 1/6$
  $P(\text{rolling an even number}) = 1/2$

If all outcomes of the experiment are equally likely then we have:

$$p = \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$$

$P(\text{not rolling a } 3) = ?$

To use

$$p = \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$$

we must first learn how to count!

- In how many different ways can the letters of the word 'hi' can be rearranged?
- What about 'factor'?

Answers: $2 = 2 \cdot 1$ and $720 = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

Do you see a pattern?

Factorial and permutations

Definition

The **factorial** of a non-negative integer n, denoted by $n!$, is the product of all positive integers less than or equal to n. In short,

$$n! := n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

Proposition

n distinct objects can be rearranged in $n!$ ways.

A menu problem

You are eating at Émile's restaurant and the waiter informs you that you have

1. two choices for appetizers: soup or juice;
2. three for the main course: a meat, fish, or vegetable dish; and
3. two for dessert: ice cream or cake.

How many possible choices do you have for your complete meal?

Answer: $2 \cdot 3 \cdot 2$

Initials problem

Prove that at least two people in Atlanta, Georgia (0.5 million inhabitants) , have the same initials, assuming no one has more than four initials.

Answer: $26 + 26 \cdot 26 + 26 \cdot 26 \cdot 26 + 26 \cdot 26 \cdot 26 \cdot 26 = 475254 < 0.5 million$

Ordered sets with repetitions

The number of k objects that can be chosen from n objects (where the order matters, and where repetitions are allowed) is

$$n^k.$$

If we don't allow repetitions, then the number is

$$n \cdot (n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}$$

In how many different ways can the letters of the word BOOKKEEPER be rearranged? (For example, the letters in the word EYE can be rearranged in three ways: EEY, EYE, and YEE.)

Answer: $\frac{10!}{2!2!3!}$ (length is 10, there are two O's, two K's and 3 E's)

Ordered vs Unordered sets

- How many different ordered pairs of people can be chosen from a group of five people?
- How many different unordered pairs of people can be chosen from a group of five people?
- How many different subsets of k people (where the order doesn't matter, and where repetitions are not allowed) can be chosen from a group of n people?

Binomial coefficient

The number of sets of k objects that can be chosen from n objects (where the order doesn't matter, and where repetitions are not allowed) is

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

usually read as "n choose k" and they are also known as binomial coefficients.

We also have

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

A card problem

Suppose that you draw a seven-card hand at random from a standard deck of 52 cards. What is the probability that your hand contains three of one card and four of another (for example: 3,3,3,3,Q,Q,Q or 6,6,6,9,9,9,9)?

Answer: First we'll count the number of seven-card pokers hands that meet this criterion. To choose such a hand, you have to specify

- which value of card to have four of, (13)
- which value of card to have three of, and (12)
- which suit is missing from the value you have three of. (4)

Since there are a total of $\binom{52}{7}$ possible seven-card hands, the probability of drawing such a hand is

$$\frac{13 \cdot 12 \cdot 4}{\binom{52}{7}}$$

Unordered sets, repetitions allowed (Section 1.7)

Pick k letters (with replacement, and with the order not mattering) from a hat containing n letters: A, B, C, ... How many different sets of k letters are possible?

- $k = 2$ and $n = 26$.
  $(26 \cdot 25)/2 + 26 = 351$
- $k = 4$ and $n = 3$.
  $3 + 6 + 3 + 3 = 15$
- $k = 3$ and $n = 4$
  $4 + 12 + 4 = 20$
- $k = 5$ and $n = 3$
  $3 + 6 + 3 + 3 = 21$

Do you see a pattern? $351 = \binom{2+25}{25}, 15 = \binom{4+2}{2}, 20 = \binom{3+3}{3}, 21 = \binom{5+2}{2}$

Unordered sets, repetitions allowed

Claim

There are $\binom{k+(n-1)}{n-1}$ ways of picking k objects (with replacement, and with the order not mattering) from a hat containing n objects.

Can you prove it?

See Section 1.7 and
`https://en.wikipedia.org/wiki/Stars_and_bars_(combinatorics)`

Resources

- Tutorial: 7-8:30 pm on Thursdays
- Office hours: M: 5:30 - 7+ pm, Th: 8 - 9 am, and by appointment
- Study groups: TBA
  Students can begin registering for Study Groups on Sunday, September 17th by accessing: `studygroups.dartmouth.edu` *and logging in with their netID and password.*

Combinatorics summary

Number of ways of picking k objects from n

- ordered, with repetition: n^k
- ordered, without repetition: $\frac{n!}{(n-k)!}$
- unordered, without repetition: $\binom{n}{k} := \frac{n!}{k!(n-k)!}$
- unordered, with repetition: $\binom{k+(n-1)}{n-1}$

Example

You have ten one-dollar bills that you want to divide among four people. How many different ways can you do this?
(The dollar bills are identical, but the people are not. So, for example, 4,0,3,3 is different from 3,4,0,3.)

Groups of people

- A group of ten people are divided into three committees. Three people are on committee A, two are on committee B, and five are on committee C. How many different ways are there to divide up the people?
  $10!/(3!2!5!)$ = number of ways of rearranging *AAABBCCCCC*
- A group of N people are divided into k committees. n_1 people are on committee 1, n_2 people are on committee 2, ..., and n_k people are on committee k, with $n_1 + n_2 + \cdots + n_k = N$. How many different ways are there to divide up the people? This called a multinomial coefficient:

$$\frac{N!}{n_1!n_2!\cdots n_k!}$$

- In how many ways can a group of 15 people be split in three groups of five?
  $\binom{15}{5}\binom{10}{5}\binom{5}{5}/3!$ or $\binom{14}{4}\binom{9}{4}\binom{4}{4}$

The Birthday Problem (M: Section 2.4.1 or GS: Example 3.3)

How many people do we need to have in a room to make it a favorable bet (probability of success greater than 1/2) that at least two people in the room will have the same birthday?

If there are n people in the room, what is the probability that all of them have different birthday?

$$\frac{365 \cdot 364 \cdot 363 \cdots (365 - (n-1))}{365 \cdot 365 \cdot 365 \cdots 365}$$

n	2	10	20	22	23	30	40
P(all different birthdays)	0.997	0.883	0.589	0.524	0.493	0.294	0.109

Independent events

Definition

Two events A and B are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

Example

Suppose that we have a coin which comes up heads with probability p, and tails with probability q. Let A be the event that heads turns up on the first toss and B the event that tails turns up on the second toss. Assuming the independence between the first and second toss, are the events A and B independent?