NAME : Key

Math 20

Midterm 1
July 14, 2017

Prof. Pantone

INSTRUCTIONS: This is a closed book exam and no notes are allowed. You are not to provide or receive help from any outside source during the exam except that you may ask the instructor for clarification of a problem. You have 120 minutes and you should attempt all problems.

- Print your name in the space provided.
- Calculators or other computing devices are not allowed.
- Except when indicated, you must show all work and give justification for your answer. **A correct answer with incorrect work will be considered wrong.**

All work on this exam should be completed in accordance with the Dartmouth Academic Honor Principle.

TIPS:

- You don't have numerically expand all answers. For example, you can leave an answer in the form $10! \cdot \binom{5}{3}^2$, rather than 362880000.
- Use scratch paper to figure out your answers and proofs before writing them on your exam.
- Work cleanly and neatly; this makes it easier to give partial credit.

Problem	Points	Score
1	24	
2	20	
3	10	
4	16	
5	10	
6	10	
7	10	
8	0	
Total	100	

Section 1: True/False.

3pts each

1. (24) Choose the correct answer. *No justification is required for your answers. No partial credit will be awarded.*

(a) If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

true for <u>all</u> A,B

True False

(b) If A and B are independent events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

true for <u>all</u> A,B

True False

(c) If A and B are mutually exclusive events, then $P(A \cap B) = P(A)P(B)$.

$P(A \cap B) = 0$ if M.E.

True False

(d) If A and B are independent events, then $P(A \cap B) = P(A)P(B)$.

True False

(e) The number of ways to line up n people in a row is n^n.

It's $n!$

True　　　　False

(f) For any sets A and B, it must be true that $A \subseteq (A \setminus B)$.

True　　　　False

(g) For any sets A and B, it must be true that $(A \setminus B) \subseteq \overline{B}$.

True　　　　False

(h) The expected value of a random variable is the numerical outcome that is most likely to occur.

True　　　　False

Section 2: Fill in the blank.

2. (20) *No justification is required for your answers. No partial credit will be awarded.*

(a) There are 100 United States senators, two from each state. If a random group of 50 senators is selected, what is the probability that there will be exactly one from each state?

# of ways to pick one senator from each state: 2^{50}

# of ways to pick 50 senators from 100: $\binom{100}{50}$

$$\text{prob} = \boxed{\frac{2^{50}}{\binom{100}{50}}}$$

(b) Suppose that you roll four fair six-sided dice simultaneously. What is the probability that together they form a straight: either $\{1,2,3,4\}$, $\{2,3,4,5\}$, or $\{3,4,5,6\}$?

There are 24 ways to get 1,2,3,4

— — — — — — 2,3,4,5

— — — — — — 3456

= $24 \cdot 3 = 72$ straights total.

$$\text{prob} = \frac{72}{6^4} = \boxed{\frac{1}{18}}$$

okay to leave your answer like this!

(c) Let A and B be independent events such that $P(A) = 0.2$ and $P(B) = 0.3$. What is $P(A \cup B)$?

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$= P(A)P(B)$ because independent.

$$= 0.2 + 0.3 - (0.2)(0.3)$$
$$= 0.5 - 0.06$$
$$\boxed{= 0.44}$$

(d) A couple decides to have children until they either have three children of the same gender, or until they have four children total, whichever comes first. Find the expected number of children they will have.

Possible outcomes:

		prob
3 children:	BBB	1/8
	GGG	1/8
4 children	2 Bs, 2 Gs $\binom{4}{2} = 6$ ways	6/16
6 more ways	BBGB BGBB GBBB ~~[illegible]~~ GGBG GBGG BGGG	6/16.

Check that $\frac{6}{16} + \frac{6}{16} + \frac{2}{8} = 1$ ✓

Expected value

$$3 \cdot \left(\tfrac{1}{4}\right) + 4 \cdot \left(\tfrac{3}{4}\right)$$
$$= \tfrac{3}{4} + 3 = \boxed{3.75 \text{ children}}$$

Section 3: Free Response.

You must show all work to receive credit.

If you need more space you may use the back of the page. You must clearly indicate on the front of the page that there is more work on the back of the page. Please work neatly.

3. (10) If Juliet gets at least 8 hours of sleep the night before class, then she manages to stay awake for the whole lecture 80% of the time. If Juliet gets between 6 and 8 hours of sleep the night before, she stays awake for the whole lecture 60% of the time. If Juliet gets less than 6 hours of sleep, she only stays awake for the whole lecture 10% of the time.

On any given night, there is a 20% probability that Juliet gets at least 8 hours of sleep, a 60% probability that she gets between 6 and 8 hours of sleep, and a 20% probability that she gets less than 6 hours of sleep.

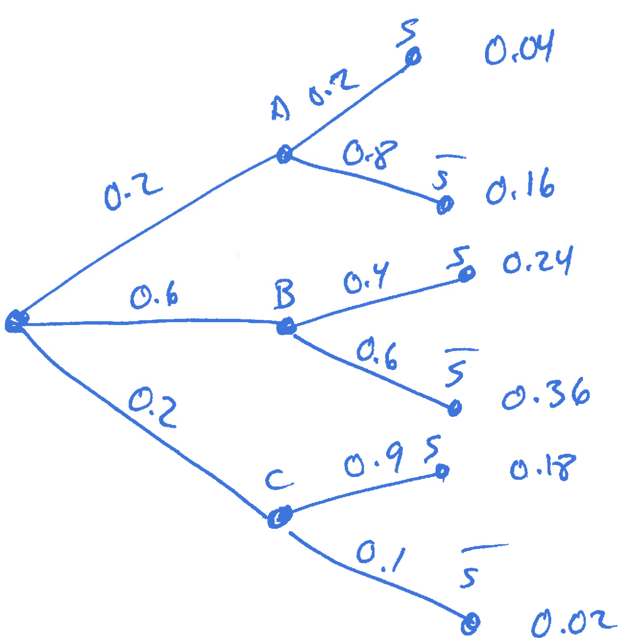
If Juliet falls asleep in class today, what is the probability that she got *at least* ***6*** *hours of sleep* the night before?

$A = \geq 8$ hours sleep

$B = < 8$ but ≥ 6 hours sleep

$C = < 6$ hours sleep

$S =$ falls asleep in class



$P(\geq 6 \text{ hours sleep} \mid \text{falls asleep})$

$= P(A \cup B \mid S)$

$= \dfrac{P((A \cup B) \cap S)}{P(S)}$

$= \dfrac{P(A \cap S) + P(B \cap S)}{P(S)}$

$= \dfrac{0.04 + 0.24}{0.04 + 0.24 + 0.18} = \dfrac{28}{46} = \boxed{\dfrac{14}{23}}$

4. (16) A typical deck of cards contains 52 cards, with 13 cards of each suit (the four suits are $\heartsuit$, $\diamondsuit$, $\spadesuit$, $\clubsuit$). This question asks about the probability of drawing different poker hands when picking 5 cards randomly from a shuffled deck.

The *order* of the cards of a suit is: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K.

If you have questions about what constitutes a deck of cards or about what the different poker hands mean, please ask me.

(a) A *flush* is a hand consisting of five cards that all have the same suit. For example, $\{3\spadesuit, 7\spadesuit, 8\spadesuit, J\spadesuit, Q\spadesuit\}$ is a flush. Let P_{flush} be the probability that a randomly drawn five-card poker hand is a flush. Find P_{flush}.

# of flushes: to build a flush, you get ~~to~~ pick one of the four suits and 5 out of 13 values.

$$\Rightarrow \binom{13}{5} \cdot 4$$

# of total hands: $\binom{52}{5}$

$$P_{\text{flush}} = \boxed{\frac{\binom{13}{5} \cdot 4}{\binom{52}{5}}}$$

(b) A *straight* is a hand consisting of five cards of consecutive value (see the "order" above). The Ace is allowed to act as either the lowest card (below 2), or the highest card (above K). The suits of the cards are irrelevant. Here are, for example, three different straights:

$$\{A, 2, 3, 4, 5\}$$
$$\{8, 9, 10, \text{J}, \text{Q}\}$$
$$\{10, \text{J}, \text{Q}, \text{K}, \text{Ace}\}$$

Let P_{straight} be the probability that a randomly drawn five-card poker hand is a straight. Find P_{straight}.

# of straights: to build a straight, you get to pick: (i) which type of straight
(ii) the suits of the 5 cards

Types of straights:

1 A 2 3 4 5
2 2 3 4 5 6
3 3 4 5 6 7
4 4 —— 8
5 5 —— 9
6 6 —— 10
7 7 —— J
8 8 —— Q
9 9 —— K
10 10 —— A

10 types.

So, # straights:

$$10 \cdot (4^5)$$

prob: $\boxed{\dfrac{10 \cdot 4^5}{\binom{52}{5}}}$

(c) A poker hand can be both a straight and a flush simultaneously (this is called a *straight flush*). Find the probability Q that a poker hand is a straight, but *not* a straight flush.

To make sure a straight is not also a flush, we have to remove (subtract) cases where all 5 cards have the same suit.

$\#: \quad 10 \cdot (4^5 - 4)$ ← remove the 4 case where all suits the same

prob $\boxed{\dfrac{10 \cdot (4^5 - 4)}{\binom{52}{5}}}$

(d) Let P_{sad} be the probability that your five-card hand has absolutely no poker value. This means: it's not a straight, not a flush, and no two of your cards have the same value (i.e., you have no *pair*, so all five values of the cards are different). Calculate P_{sad}.

To make a sad hand, you need to pick 5 card values out of 13 that are all different (or else you have a pair).

This allows for straights, so we subtract the 10 value choices that make straights.

Then, we pick any combination of suits, subtracting the 4 cases that make flushes.

$\boxed{\text{Prob} = \dfrac{\left(\binom{13}{5} - 10\right)(4^5 - 4)}{\binom{52}{5}}}$

5. (10) Let Ω be a finite sample space and let A, B, and C be events. Prove that

$$P(A)P(B|A)P(C|A\cap B) = P(A\cap B\cap C).$$

(Do not use tree diagrams in your proof.)

By definition, $P(B|A) = \dfrac{P(B\cap A)}{P(A)}$

and

$P(C \mid A\cap B) = \dfrac{P(C\cap(A\cap B))}{P(A\cap B)}$.

So, $P(A)P(B|A)P(C|A\cap B) =$

$\cancel{P(A)} \cdot \dfrac{\cancel{P(B\cap A)}}{\cancel{P(A)}} \cdot \dfrac{P(C\cap A\cap B)}{\cancel{P(A\cap B)}}$

$= P(A\cap B\cap C)$. $\boxed{\text{B}}$

6. (10) Karen is a pretty good tennis player, and her friend, Henry, offers her \$100 if she can complete the following challenge. To win the money, she has to win two tennis matches in a row out of three total. (Just to clarify, this means she gets the money if she wins Matches 1 and 2, or if she wins Matches 2 and 3, but she does not get the money if she only wins Matches 1 and 3.)

In each of her games, she has to play against either her friend Henry (`H`) or against the club champion Carlton (`C`). The options are to play Henry in the first and third games and Carlton in the middle (`H,C,H`) or to play Carlton in the first and third games and Henry in the middle (`C,H,C`).

Assume that Carlton is a better tennis player than Henry, and that the outcome of each match is independent of the outcomes of the previous matches. Which of the two match orders—`CHC` or `HCH`—gives Karen a better chance at winning the \$100?

(*Hint:* Start by setting p to be the probability that Karen beats Carlton in any given match and q to be the probability that Karen beats Henry in any given match. We're assuming $p < q$.)

We will calculate the probability that Karen wins 2 in a row for each option, then compare.

CHC:

w/l	prob
wwl	$pq(1-p)$
lww	$(1-p)qp$
www	pqp

So, $P(\text{win } \$) = 2pq(1-p) + p^2 q$

HCH:

w/l	
wwl	$qp(1-q)$
lww	$(1-q)pq$
www	qpq

So, $P(\text{win } \$) = 2pq(1-q) + pq^2$.

Now the question is which is bigger: $2pq(1-p)+p^2q$ or $2pq(1-q) + pq^2$

Notice that $p \cdot q > 0$, so dividing both by $p \cdot q$ won't affect which is bigger:

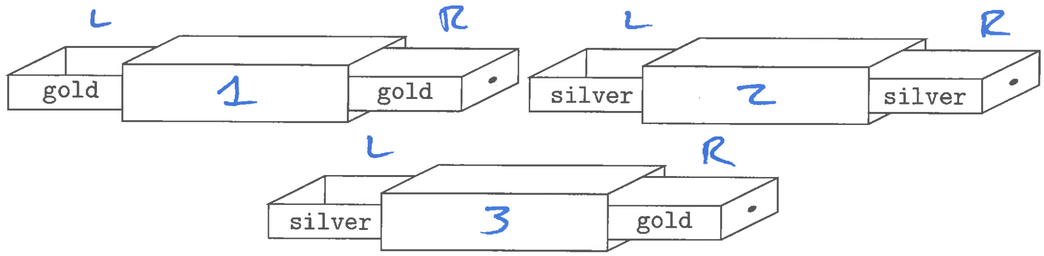
$$2(1-p) + p \quad \text{vs.} \quad 2(1-q) + q$$
$$\quad \| \qquad\qquad\qquad\qquad \|$$
$$2-p \qquad\qquad\qquad 2-q.$$

Since $p < q$, $2-p$ is bigger than $2-q$.

→ Therefore $\boxed{CHC}$ is better.

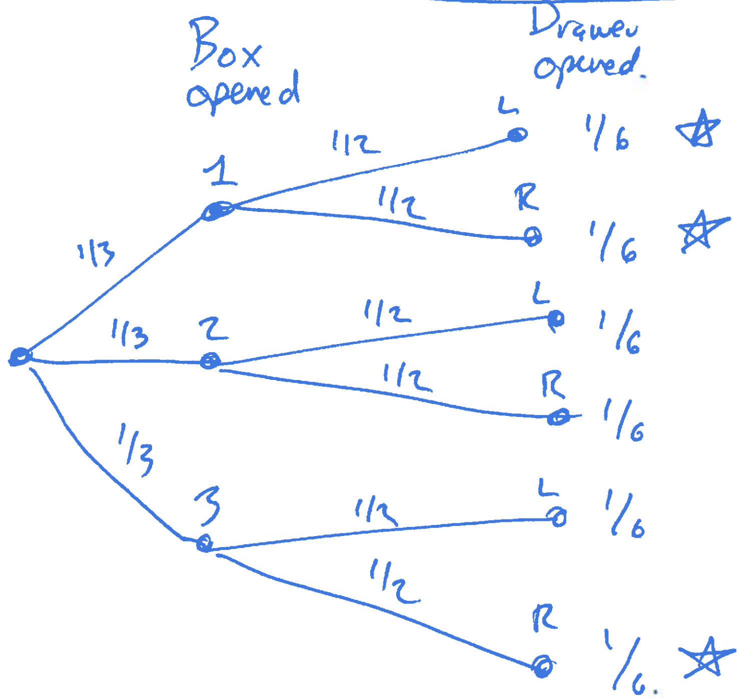
Counterintuitive!

7. (10) Buzz has three wooden boxes, and each box has two drawers, one on each side. Buzz takes three gold rings and three silver rings and distributes one into each drawer: one box gets gold rings in both drawers, one gets silver rings in both drawers, and one gets one gold and one silver. See the picture below.



Now, Woody comes in the room and randomly picks one of the three boxes, and randomly opens one of the two drawers in the box. If that drawer contains a gold ring, then what is the probability that the other drawer also contains a gold ring?

the boxes 1, 2, 3 as above, and the drawers L, R



If one drawer is opened and it's gold, then we're in one of the 3 ☆ situations.

Of those, the top 2 result in the other drawer having a gold ring.

So, prob $= \dfrac{\frac{1}{6}+\frac{1}{6}}{\frac{1}{6}+\frac{1}{6}+\frac{1}{6}} = \boxed{\dfrac{2}{3}}$

8. (0) **Bonus:** *(5 points)* Shawn and Gus each flip a fair coin n times. What is the probability that they both flip heads the same number of times?

This one is tricky. That's why it's a bonus Q!

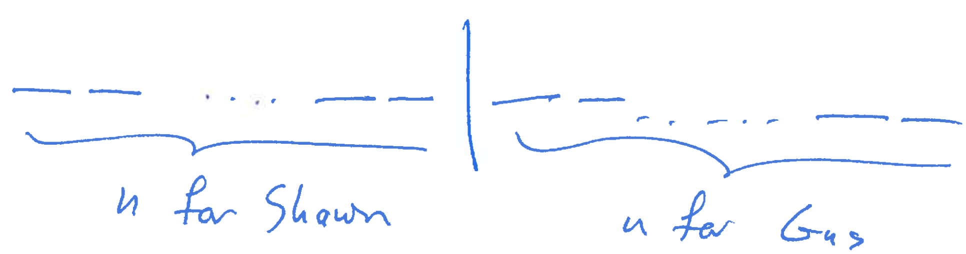
Fact: If you flip a coin N times, then

prob(k heads) = prob($N-k$ heads)

for all k. (Do you see why?)

So, we'll calculate the probability that Shawn flips k heads and Gus flips $n-k$ heads, for some k.

Picture the $2n$ total coin flips as blanks



Saying that Shawn flips k heads and Gus flips $n-k$ heads for some k is only saying that of these $2n$ blanks, n are heads (and thus n are tails.)

The # of ways to fill in $2n$ blanks with n heads and n tails is $\binom{2n}{n}$. The total # of possible coin flip outcomes is $2^{(2n)}$. So, prob = $\boxed{\dfrac{\binom{2n}{n}}{4^n}}$