NAME: ~ANSWER KEY~

Math 20

Summer 2015

Exam I

Instructions:

1. Write your name *legibly* on this page.
2. There are nine problems, some of which have multiple parts. Do all of them.
3. Explain what you are doing, and show your work. You will be *graded on your work*, not just on your answer. Make it clear and legible so I can follow it.
4. It is okay to leave your answers unsimplified. That is, if your answer is the sum or product of 5 numbers, you do not need to add or multiply them. Answers left in terms of binomial coefficients or factorials are also acceptable. However, do not leave any infinite sums or products, or sums or products of a variable number of terms.
5. There are a few pages of scratch paper at the end of the exam. I *will not look* at these pages unless you write on a problem "Continued on page..."
6. This exam is closed book. You may not use notes, calculators, or any other external resource. It is a violation of the honor code to give or receive help on this exam.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total	90	

1. (10 points.) Let X be a continuous random variable. Suppose that the density function for X is given by:

$$f(t) = \begin{cases} 0 & \text{if } t < -1 \\ c(t+1)^2 & \text{if } -1 \le t \le 0 \\ c(t-1)^2 & \text{if } 0 < t \le 1 \\ 0 & \text{if } t > 1 \end{cases}.$$

(a) Find c.

(b) Compute the cumulative distribution function for X.

(a) MUST HAVE $2c\int_{-1}^{0} (t+1)^2\,dt = 2\frac{c}{3}(t+1)^3\Big|_{-1}^{0} = \frac{2}{3}c = 1$

SO $c = \frac{3}{2}$.

(b) $F(z) = \int_{-\infty}^{z} f(t)\,dt$

$$= \begin{cases} 0 & \text{IF } z < -1 \\ \int_{-1}^{z} f(t)\,dt & \text{IF } -1 \le z \le 0 \\ \int_{-1}^{0} f(t)\,dt + \int_{0}^{z} f(t)\,dt & \text{IF } 0 < z \le 1 \\ 1 & \text{O.W.} \end{cases}$$

$$= \begin{cases} 0 & \text{IF } z < -1 \\ \frac{1}{2}(z+1)^2 & \text{IF } -1 \le z \le 0 \\ \frac{1}{2} + \left[\frac{1}{2}(z-1)^3 + \frac{1}{2}\right] & \text{IF } 0 < z \le 1 \\ 1 & \text{O.W.} \end{cases}$$

2. (10 points.) If $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{2}$, and $P(A \cup B) = \frac{3}{4}$, find each probability:

(a) $P(A \cap B)$

(b) $P(\tilde{A} \cup \tilde{B})$

(c) $P(\tilde{A} \cap B)$

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\frac{3}{4} = \frac{1}{3} + \frac{1}{2} - P(A \cap B)$$

$\therefore \quad P(A \cap B) = \frac{4}{12} + \frac{6}{12} - \frac{9}{12} = \frac{1}{12}$

(b) $\tilde{A} \cup \tilde{B} = \widetilde{(A \cap B)}$ (DRAW A PICTURE)

SO $P(\tilde{A} \cup \tilde{B}) = 1 - P(A \cap B) = \frac{11}{12}$

(c) $B = \tilde{A} \cap B \overset{\text{DISJOINT UNION}}{\cup} A \cap B$

SO $P(B) = \frac{1}{2} = P(\tilde{A} \cap B) + P(A \cap B)$

$$= P(\tilde{A} \cap B) + \frac{1}{12}$$

$\therefore \quad \frac{5}{12} = P(\tilde{A} \cap B).$

3. (10 points.) An urn contains n red and m blue balls. They are withdrawn one at a time (without replacement) until all the red balls have been withdrawn. Find the probability that a total of k balls are drawn where $n \leq k \leq n+m$.

THERE ARE A TOTAL OF $\binom{m+n}{n}$ WAYS OF CHOOSING n RED BALLS FROM THE URN. FOR A FIXED k THERE ARE $\binom{k-1}{n-1}$ WAYS OF PICKING $n-1$ RED BALLS IN THE FIRST $k-1$ SLOTS. WE KNOW THE FINAL BALL MUST BE RED SO JUST 1 CHOICE FOR THE k^{th} BALL.

MANY OF YOU THOUGHT THIS WAS A BERNOULLI TRIALS PROCESS. IT'S NOT SINCE THE BALLS ARE NOT BEING REPLACED.

THUS THE PROBABILITY THAT A TOTAL OF k BALLS ARE DRAWN IS:

$$\frac{\binom{k-1}{n-1}}{\binom{m+n}{n}}.$$

4. (10 points.) Suppose the average length of a phone call is 10 minutes, and we model the duration of a call using the exponential density with average 10. Someone calls you.

   (a) Find the probability that your phone call is more than 10 minutes.

   (b) Your favorite TV show begins in 15 minutes. Given that the call is still in progress when the show starts, what is the probability that you miss more than the first five minutes of your show.

(a) HERE $\lambda = \frac{1}{10}$. SO WE HAVE

$$P(T > 10) = \int_{10}^{\infty} \frac{1}{10} e^{-\frac{t}{10}}\, dt = -e^{-\frac{t}{10}} \Big|_{10}^{\infty} = e^{-1}$$

(b) THIS IS A CONDITIONAL PROBABILITY PROBLEM. WE WANT TO COMPUTE:

$$P(T > 20 \mid T > 15) = \frac{P(\{T > 20\} \cap \{T > 15\})}{P(T > 15)}$$

$$= \frac{\int_{20}^{\infty} \frac{1}{10} e^{-\frac{t}{10}}\, dt}{\int_{15}^{\infty} \frac{1}{10} e^{-t/10}\, dt}$$

$$= \frac{e^{-2}}{e^{-\frac{3}{2}}} = e^{-1/2}.$$

5. (10 points.) What is the probability that a randomly chosen integer from 1 to 1000 is not divisible by 2, 7, or 9?

SET $a_n = \frac{1}{1000} \cdot \left\lfloor \frac{1000}{n} \right\rfloor$. THIS IS THE PROBABILITY THAT

FLOOR FUNCTION

A RANDOM INTEGER IS DIVISIBLE BY n.

LET A_n BE THE EVENT "IS DIVISIBLE BY n." THEN $P(A_n) = a_n$.

WE WANT TO COMPUTE $1 - P(A_2 \cup A_7 \cup A_9)$.

$$P(A_2 \cup A_7 \cup A_9) = P(A_2) + P(A_7) + P(A_9) - \left(P(A_2 \cap A_7) + P(A_2 \cap A_9) + P(A_7 \cap A_9)\right)$$
$$+ P(A_2 \cap A_7 \cap A_9).$$

(BY THE LAW OF INCLUSION / EXCLUSION).

NOW $P(A_i \cap A_j) = P(A_{\mathrm{LCM}(i,j)}) = a_{\mathrm{LCM}(i,j)}$

SO $P(A_2 \cap A_7) = a_{14}$, $P(A_2 \cap A_9) = a_{18}$, $P(A_7 \cap A_9) = a_{63}$.

SIMILARLY $P(A_2 \cap A_7 \cap A_9) = a_{126}$.

6. (10 points.) Alice has two coins in her pocket, a fair coin and a two-headed coin. She picks one randomly from her pocket and tosses it. The coin comes up heads. What is the probability that she tossed the two-headed coin?

$$P(U|H) = \frac{P(U \cap H)}{P(H)}$$

$$= \frac{1/2}{P(H \cap U) + P(H \cap F)} = \frac{1/2}{1/2 + 1/4} = \frac{1/2}{3/4}$$

$$= \frac{2}{3}.$$

7. (10 points.) The Mariners and the Phillies are in the World Series. The World Series is a series of up to seven games where the first team to win 4 games wins the series. The Phillies, being the more talented team, win each game with probability $\frac{3}{4}$. What is the probability that the Phillies win the series in 6 or more games?

THE PROBABILITY THAT THE PHILLIES WIN IN n GAMES WHERE $4 \le n \le 7$

$= P($PHILLIES WIN 3 OF FIRST $n-1$ GAMES $\cap$ PHILLIES WIN n^{th} GAME$)$

$= P($Ps WIN 3 OF FIRST $n-1$ GAMES$)P($Ps WIN n^{th} GAME$)$

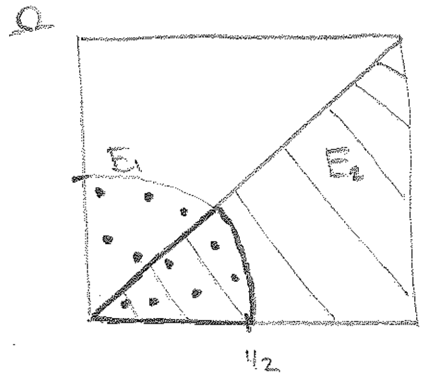
BY INDEPENDENCE

$$= \binom{n-1}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^{n-1-3} \cdot \left(\frac{3}{4}\right) = \binom{n-1}{3}\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^{n-1-3}$$

THIS IS A BERNOULLI TRIALS PROCESS.

SO $P($PHILS WIN IN 6 OR 7$) = \binom{5}{3}\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^2 + \binom{6}{3}\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^3.$

8. (10 points.) Let x and y be chosen at random from $[0, 1]$. Let E_1 be the event $x^2+y^2 \leq \frac{1}{4}$ and let E_2 be the even $x \geq y$. Are E_1 and E_2 independent?



$P(E_1 \cap E_2) = \frac{1}{2} \cdot \frac{1}{4}\pi \cdot \frac{1}{4} = \frac{\pi}{32}$

$P(E_1) = \frac{\pi}{16}$

$P(E_2) = \frac{1}{2}$.

SINCE $P(E_1)P(E_2) = P(E_1 \cap E_2)$, E_1 & E_2 ARE INDEPENDENT.

9. (10 points.) Show the following identity:

$$\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}.$$

[Hint: One way of doing this is by showing that both quantities count the number of ways of determining a committee of any size from a collection of n people and assigning a chairperson for the committee.]

LHS: FIX $k \geq 1$ THE NUMBER OF FOLKS YOU WANT IN THE COMMITTEE. THERE ARE $\binom{n}{k}$ CHOICES FOR PEOPLE IN THE COMMITTEE. OUT OF THE FOLKS IN THE COMMITTEE THERE ARE k CHOICES FOR THE CHAIRPERSON. SUMMING OVER k GIVES ALL THE POSSIBILITIES.

RHS: PICK A CHAIRPERSON FIRST. YOU HAVE n CHOICES. NOW YOU CAN ADD AN ARBITRARY SUBSET OF THE $n-1$ OTHER FOLKS TO FORM A COMMITTEE W/ CHAIRPERSON. THERE ARE 2^{n-1} SUBSETS TO CHOOSE. SO THERE ARE A TOTAL OF $n \cdot 2^{n-1}$ POSSIBILITIES.