1. SHORT ANSWER:

Answer each of the following questions. You do not need to show any work, but can be assessed partial credit on work shown. (3 points each)

(a) [3 points] Let A and B be independent events with $P(A) = .4$ and $P(B) = .5$. What is $P(A \cup B)$?

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

$$= .4 + .5 - \mathbb{P}(A)\mathbb{P}(B) \leftarrow \text{ind.}$$

$$.4 \cdot .5$$

$$= .9 - .2 = \boxed{.7}$$

(b) [3 points] A trip leader has decided to paint all of the fingernails on his right hand using exactly two colors. If he access to four colors of nail polish, how many ways can he do this?

$$\binom{4}{2}(2^5 - 2)$$

colors

use 2 on the five finger nails

(c) [3 points] You are dealt six cards from a 52 card deck. What is the probability exactly four of your cards are of the same suit?

cards from suit

$$\frac{4\binom{13}{4}\binom{39}{2}}{\binom{52}{6}} \leftarrow \text{choose remaining cards}$$

suit

(d) [3 points] Volleyball games are played to 25. My team wins each point with probability .7. Find an expression for the probability my team wins by eight.

Neg. binomial

$$\underbrace{\binom{41}{24}}_{\text{first 24 pts}}(.7)^{25}(.3)^{17}$$

(e) [3 points] Having broken out of your cell and stolen the warden's set of keys, you are one door away from getting out of prison. Assume there are n keys total, each equally likely to be the correct key. How many do you expect to have to try before finding the right key (you can keep track of which keys you have already used)?

$\frac{n+1}{2}$ Uniform on $\{1,2,\ldots,n\}$ expected value

(correct key is unif.)

(f) [3 points] In your frantic rush to the outside world, you fail to keep track of which keys you try on the last door. In this case, how many keys do you expect to try (there are still n keys)?

n geometric$\left(\frac{1}{n}\right)$ expected value

2. April, a graduate student in the Dartmouth Math Department, is preparing to take a qualifying exam (not multiple choice) in Algebra. There is a list of 75 questions posted on the department website. The exam consists of 5 of these questions, of which she must answer four correctly in order to pass. April goes into the exam knowing how to answer 65 of the questions.

   (a) [6 points] On average, how many questions will April answer correctly. (This value should **not** contain a sum.)

$X = \#$ correct

$$\mathbb{E}(X) = \underbrace{\frac{65}{75}}_{\text{prob a Q is}} \cdot \underbrace{5}_{\# Q} = \frac{65}{15} = \frac{13}{3}$$

   (b) [6 points] Find the exact probability that April passes (you do not need to simplify your answer).

Hypergeometric

$$\underbrace{\frac{\binom{65}{4}\binom{10}{1}}{\binom{75}{5}}}_{4 \text{ correct}} + \underbrace{\frac{\binom{65}{5}}{\binom{75}{5}}}_{5 \text{ correct}}$$

3. [12 points] There are two urns. The first has one red ball and two white balls, while the second has two red balls and one white ball. One of the urns is selected at uniformly at random. If you guess which urn it is correctly, you get $10,000. You can have balls drawn from the urn, with replacement, at the cost of decreasing your prize. For $750, you can draw one ball (so if you win, you only get $9250). For $1000, you can draw two balls. What strategy should you use? Justify your answer.

X = winnings

No draws:

$\mathbb{E}(X) = \frac{1}{2}\,10000 = 5000$

1 Draw: ☆Best☆

Majority $2/3$

Minority $1/3$

so $\mathbb{E}(X) = 9250 \cdot \frac{2}{3} + 0 \cdot \frac{1}{3} = \frac{18,500}{3} = 6166\frac{2}{3}$

2 Draw: (w/ replacement)

Both majority: $4/9$

One of each: $2 \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{9}$

Both minority: $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$

guess right

so $\mathbb{E}(X) = 9000 \cdot \frac{4}{9} + 9000 \cdot \frac{1}{2} \cdot \frac{4}{9} + 0 \cdot \frac{1}{9}$

$= 4000 + 2000 = 6000.$

4. We are presented with a perfectly shuffled deck of cards. Looking through the deck, we notice that there are two fives next to each other.

   (a) [6 points] On average, how many such pairs exist?

$$\mathbb{P}(i, i+1 \text{ are pair}) = \frac{52}{52} \cdot \frac{3}{51} = \frac{1}{17}$$

ith card is whatever

(i+1)th card same value

Can have a pair for $i = 1, 2, \dots, 51$

so $\mathbb{E}(\#\text{ pairs}) = \frac{51}{17} = 3$

   (b) [6 points] **Estimate** the probability that such a pair exists.

Poisson approx

$\lambda = 3$

$\mathbb{P}(1+\text{ pair}) = 1 - \mathbb{P}(\text{no pair}) \approx 1 - e^{-3}$

(could use binomial, but not ind., so note is an approx)

5. [6 points] Let X be a random variable with mean μ and variance σ^2. Find the mean and variance of

$$\frac{X-\mu}{\sigma}.$$

$$\mathbb{E}\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma}\mathbb{E}(X-\mu) = \frac{1}{\sigma}\big(\underbrace{\mathbb{E}(X)-\mu}_{0}\big)$$
$$= 0$$

$$V\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}V(X-\mu) = \frac{1}{\sigma^2}V(X)$$
$$= \frac{1}{\sigma^2}\sigma^2 = 1$$

6. DIETING PROBLEMS: I want to know how much I weigh. I have two scales saying I weigh S_1 and S_2 pounds, respectively. Each scale's average is my true weight μ, but they are inaccurate with variances $V(S_1) = \sigma_1^2$ and $V(S_2) = \sigma_2^2$. We will use a weighted average

$$X_w = wS_1 + (1-w)S_2$$

with $0 < w < 1$ to estimate my weight.

(a) [4 points] What is $\mathbb{E}(X_w)$?

$$\mathbb{E}(X_\omega) = \omega\,\mathbb{E}(S_1) + (1-\omega)\,\mathbb{E}(S_2)$$
$$= \omega\mu + (1-\omega)\mu = \mu$$

(b) [8 points] What value of w produces the estimate with the smallest variance?

$$V(X_\omega) = V(\omega S_1) + V((1-\omega)S_2)$$

S_1, S_2 ind.

$$= \omega^2 \sigma_1^2 + (1-\omega)^2 \sigma_2^2$$ minimize this.

$$\frac{d\,V(X_\omega)}{d\omega} = 2\omega\sigma_1^2 - 2(1-\omega)\sigma_2^2$$

2nd deriv >0 so is min.

set to 0

$$0 = 2\left(\omega(\sigma_1^2+\sigma_2^2) - \sigma_2^2\right)$$

$$\sigma_2^2 = \omega(\sigma_1^2+\sigma_2^2) \quad \text{so} \quad \boxed{\omega = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}}$$

7. [8 points] Let X and Y be independent Poisson random variables with parameters λ_X and λ_Y. Show $X+Y$ is Poisson.

$$\mathbb{P}(X+Y=k) = \sum_{j=0}^{k} \mathbb{P}(X=j, Y=k-j)$$

need $\mathbb{P}$,

$\mathbb{E}$ & $\mathbb{V}$ are evidence, but not proof

ind.

$$= \sum_{j=0}^{k} \mathbb{P}(X=j)\mathbb{P}(Y=k-j)$$

$$= \sum_{j=0}^{k} e^{-\lambda_X}\frac{(\lambda_X)^j}{j!} e^{-\lambda_Y}\frac{(\lambda_Y)^{k-j}}{(k-j)!}$$

$$= e^{-(\lambda_X+\lambda_Y)} \sum_{j=0}^{k} \frac{1}{j!(k-j)!} (\lambda_X)^j(\lambda_Y)^{k-j}$$

$$= e^{-(\lambda_X+\lambda_Y)} \frac{1}{k!} \underbrace{\sum_{j=0}^{k} \underbrace{\frac{k!}{j!(k-j)!}}_{\binom{k}{j}} (\lambda_X)^j(\lambda_Y)^{k-j}}_{\text{binomial theorem}}$$

$$= e^{-(\lambda_X+\lambda_Y)} \frac{(\lambda_X+\lambda_Y)^k}{k!}$$

so $X+Y$ is Poisson of parameter $(\lambda_X+\lambda_Y)$