NAME : Key

Math 20

Final Exam
August 27, 2017

Prof. Pantone

INSTRUCTIONS: This is a closed book exam and no notes are allowed. You are not to provide or receive help from any outside source during the exam except that you may ask the instructor for clarification of a problem. You have 180 minutes and you should attempt all problems.

- Print your name in the space provided.
- **Calculators or other computing devices are not allowed.**
- Except when indicated, you must show all work and give justification for your answer. **A correct answer with incorrect work will be considered wrong.**

All work on this exam should be completed in accordance with the Dartmouth Academic Honor Principle.

If you use facts of a distribution, you must name the distribution and justify why it's appropriate.

TIPS:

- You don't have numerically expand all answers. For example, you can leave an answer in the form $10! \cdot \binom{5}{3}^2$, rather than 362880000.
- Use scratch paper to figure out your answers and proofs before writing them on your exam.
- Work cleanly and neatly; this makes it easier to give partial credit.

Problem	Points	Score
1	24	
2	24	
3	6	
4	8	
5	8	
6	6	
7	16	
8	8	
9	0	
Total	100	

Section 1: True/False.

1. (24) Choose the correct answer. *No justification is required for your answers. No partial credit will be awarded.*

(a) If X, Y, and Z are random variables and a, b, and c are real numbers, then

$$\mathbb{E}[aX + b(Y + Z) + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + b\mathbb{E}[Z] + c.$$

Linearity of expectation

True　　　　False

(b) Let P be the transition matrix for an absorbing Markov chain. Then, $\lim_{n\to\infty} P^n$ converges to a limiting matrix.

$$\lim_{n\to\infty} P^n = \left(\begin{array}{c|c} 0 & * \\ \hline 0 & I \end{array}\right)$$

True　　　　False

(c) The number of ways to pick 5 people out of a class of 20 and sit them in a line in the front row of the classroom is $20 \cdot 19 \cdot 18 \cdot 17 \cdot 16$.

leftmost　　etc

next leftmost

True　　　　False

(d) A Markov chain can be used to model processes where the probability of transition from state i to state j at time t depends only on the states the process was in at times 1 through $t - 1$.

Can only depend on previous state (time $t-1$)

True　　　　False

(e) If A and B are independent events, then

$$P(A \mid B) = P(B).$$

If A, B ind., $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$

True False

(f) Let A and B be events. Then, $P(A \cup B) = P(A) + P(B)$ if A and B are mutually exclusive.

$P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{=0 \text{ if mutually exclusive.}}$

True False

(g) The Weak Law of Large Numbers states that if S_n is the sum of n i.i.d. random variables, then the distribution of S_n converges to a normal distribution as $n \to \infty$.

Says nothing about the distribution, just the mean.

True False

(h) The Poisson distribution is a continuous distribution.

It is discrete.

True False

(i) The Central Limit Theorem states the sum of a large number of i.i.d. random variables has a normal distribution.

It approaches or approximates a normal dist, but almost always doesn't have one.

True False

(j) Suppose that an amount of money \$X is chosen according to a distribution p_X known to you, that one envelope is filled with \$X and another with \$2X, and that you are randomly given one of the two envelopes. Then, after looking at the amount of money in your envelope, it is possible to determine whether switching envelopes increases your expected winnings.

This is the whole $p_x^* > \frac{1}{3}$ thing.

True False

(k) Let X and Y be uniform random variables on $[2, 3]$ with probability density functions f_X and f_Y and let $Z = X + Y$. Then, the probability density function f_Z of Z is

$$f_Z(z) = \int_2^3 f_X(x) f_Y(z - x) dx.$$

Formula says $\int_{-\infty}^{\infty}$, but $f_X(x)$ is 0 unless $x \in [2,3]$.

True False

(l) The standard deviation of a random variable is positive if the distribution is right-skewed and negative if the distribution is left-skewed.

standard deviation can't be negative.

True False

Section 2: Short Answer.

2. (24) $\boxed{\textit{Justify all answers.}}$

(a) Prove that if A and B are independent events then $P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$.

Proof: By the definition of independent, $P(A \cap B) = P(A)P(B)$.

By the formula for conditional probability,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

So, $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A)\cancel{P(B)}}{\cancel{P(B)}} = P(A).$

Similarly, $P(B \mid A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{P(B)\cancel{P(A)}}{\cancel{P(A)}} = P(B).$ ∎

(b) Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with $\mu = 1$ and $\sigma^2 = 2$. Let $S_n = X_1 + \cdots + X_n$ and $A_n = S_n/n$. Without referring to a z-table, determine whether each of the quantities is "equal to zero", "equal to 1", or "equal to a constant $0 < c < 1$". If you choose the last option, you *do not need to identity the actual value of* c.

$\mathbb{E}[S_n] = n \quad \mathbb{E}[A_n] = 1$

$Var(S_n) = 2n \quad Var(A_n) = \frac{2}{n}$

$\sigma_{S_n} = \sqrt{2}\sqrt{n} \quad \sigma_{A_n} = \frac{\sqrt{2}}{\sqrt{n}}$

(i) $\lim_{n\to\infty} P(0.99n < S_n < 1.01n)$

$$= \lim_{n\to\infty} P\left(\underbrace{\frac{-0.01n}{\sqrt{2}\sqrt{n}}}_{\xrightarrow{n\to\infty} -\infty} < \frac{S_n - n}{\sqrt{2}\sqrt{n}} < \underbrace{\frac{0.01n}{\sqrt{2}\sqrt{n}}}_{\xrightarrow{n\to\infty} \infty}\right)$$

$\boxed{= 1}$

(ii) $\lim_{n\to\infty} P(n + 0.99\sqrt{n} < S_n < n + 1.01\sqrt{n})$

$$= \lim_{n\to\infty} P\left(\frac{0.99\sqrt{n}}{\sqrt{2}\sqrt{n}} < \frac{S_n - n}{\sqrt{2}\sqrt{n}} < \frac{1.01\sqrt{n}}{\sqrt{2}\sqrt{n}}\right)$$

$$= \lim_{n\to\infty} P(\text{constant} < S_n^* < \text{other constant}) \boxed{= c,}$$

$0 < c < 1.$

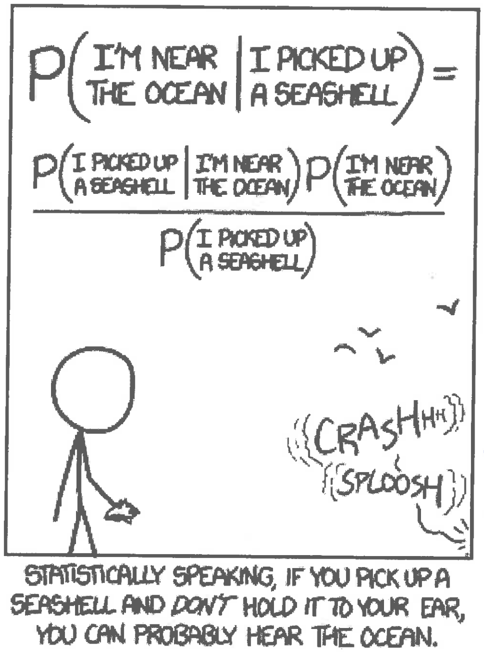
(iii) $\lim_{n\to\infty} P(A_n = 1)$

$\boxed{= 0}$ (increasingly unlikely A_n is exactly 1 as $n \to \infty$)

(iv) $\lim_{n\to\infty} P(0.99 < A_n < 1.01)$

$$= \lim_{n\to\infty} P(0.99n < S_n < 1.01n) \boxed{= 1,} \text{ as above}$$

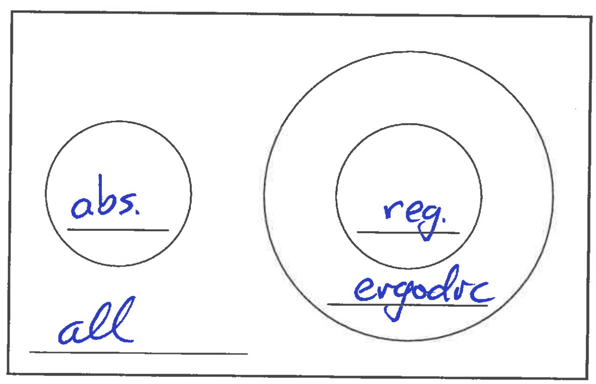
(c) What formula is being used in this comic? Explain the joke.



(credit: xkcd.com)

This is Bayes' Theorem. People always point out that you can "hear the ocean" if you hold a shell to your ear. This points out that you can probably hear the ocean even without that last step, because given that you picked up a shell, you're probably already within earshot of the ocean.

(d) Fill in the blanks in the Venn diagram below with the following labels: "*all Markov chains*", "*absorbing Markov chains*", "*ergodic Markov chains*", "*regular Markov chains*". (You can abbreviate to save room.)



(e) Six scraps of paper are put into a hat. Three are labeled with the number 5, two are labeled with the number 8, and one is labeled with the number 11. You draw one piece of paper out of the hat at random. Let X be the random variable for the value written on that paper. Find $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

$$X = \begin{cases} 5, & \text{with prob } 1/2 \\ 8, & \text{with prob } 1/3 \\ 11, & \text{with prob } 1/6. \end{cases}$$

$$\mathbb{E}[X] = 5 \cdot \frac{1}{2} + 8 \cdot \frac{1}{3} + 11 \cdot \frac{1}{6} = \frac{5}{2} + \frac{8}{3} + \frac{11}{6} = \frac{15+16+11}{6} = \frac{42}{6}$$

$$\boxed{= 7.}$$

$$(X - \mathbb{E}[X])^2 = \begin{cases} 4, & \text{with prob } \frac{1}{2} \\ 1, & \text{with prob } \frac{1}{3} \\ 16, & \text{with prob } \frac{1}{6}. \end{cases}$$

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = 4 \cdot \frac{1}{2} + 1 \cdot \frac{1}{3} + 16 \cdot \frac{1}{6} = \frac{12+2+16}{6} \boxed{= 5}$$

(f) Since 1851, exactly 116 hurricanes have hit Florida (this includes the years 1851 and 2016, but not 2017—only direct hits by *hurricanes* are counted, not tropical storms). In 2005, Florida was hit by four hurricanes: Cindy, Dennis, Katrina, and Wilma. If the probability of hurricane strikes has remained the same since 1851, what is the probability of Florida being struck by four or more hurricanes in the same year? You may leave your answer as a summation.

\# of years = 2016 − 1851 + 1 = 166.

This is a Poisson RV with rate $\lambda = 116/166$.

For a Poisson RV: $P(X=k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$

So, answer = $\displaystyle\sum_{k=4}^{\infty} \frac{\left(\frac{116}{166}\right)^k e^{-116/166}}{k!}$

or $\displaystyle 1 - \sum_{k=0}^{3} \frac{\left(\frac{116}{166}\right)^k e^{-116/166}}{k!}$

Section 3: Free Response.

You must show all work to receive credit.

If you need more space you may use the back of the page. You must clearly indicate on the front of the page that there is more work on the back of the page. Please work neatly.

3. (6) In order to test whether a batch of 100 widgets meets specifications, a manufacturer picks 20 items at random. If none are defective, the batch is accepted. If at least one is defective, the manufacturer — whose turns out to be quite unscrupulous — mixes the 20 widgets back in with the 100, then picks 20 more at random. If none are defective, the batch is accepted. If at least one is defective, then the manufacturer finally rejects the batch. If a batch of 100 items has 10 defective widgets, what is the probability that it gets accepted?

(See homework solution for more details.)

let A = first accepted
B = first rejected
C = second accepted

$$P(\text{batch accepted}) = P(A) + P(B \cap C)$$
$$= P(A) + P(\bar{A} \cap C)$$
$$= P(A) + P(\bar{A})P(C) \quad \text{(independent)}$$
$$= \frac{\binom{90}{20}}{\binom{100}{20}} + \left(1 - \frac{\binom{90}{20}}{\binom{100}{20}}\right)\left(\frac{\binom{90}{20}}{\binom{100}{20}}\right)$$
$$= 2 \cdot \frac{\binom{90}{20}}{\binom{100}{20}} - \left(\frac{\binom{90}{20}}{\binom{100}{20}}\right)^2.$$

No continuity correction.

4. (8) The price of one share of stock in the Pilsdorff Beer Company is given by Y_n on day n. You observe that the differences $X_n = Y_{n+1} - Y_n$ are continuous i.i.d. random variables with mean 0 and variance $1/4$. If $Y_1 = 100$ then use the Central Limit Theorem to estimate the following probabilities. Use the z-table at the back of this exam.

(a) $P(Y_{401} \geq 100)$

$$Y_{400} = Y_1 + X_1 + X_2 + \cdots + X_{400}.$$

let $S_{400} = X_1 + \cdots + X_{400}$.

$$E[S_{400}] = 400 \cdot 0 = 0.$$

$$Var(S_{400}) = 400 \cdot \frac{1}{4} = 100$$

$$\sigma_{S_{400}} = \sqrt{100} = 10.$$

$$= P(S_n \geq 0)$$

$$= P\left(\frac{S_n - 0}{10} \geq 0\right)$$

$$\boxed{\approx 0.5.}$$

(b) $P(Y_{401} \geq 115)$

$$= P(S_n \geq 15) = P\left(\frac{S_n}{10} \geq \frac{15}{10}\right)$$

$$= P\left(\frac{S_n}{10} \geq 1.5\right)$$

$$\approx 1 - 0.9332$$

$$\boxed{= 0.0668.}$$

(c) $P(Y_{401} \geq 78)$

$$= P(S_{400} \geq -22) = P\left(\frac{S_{400}}{10} \geq -2.2\right)$$

$$\approx 1 - 0.0139$$

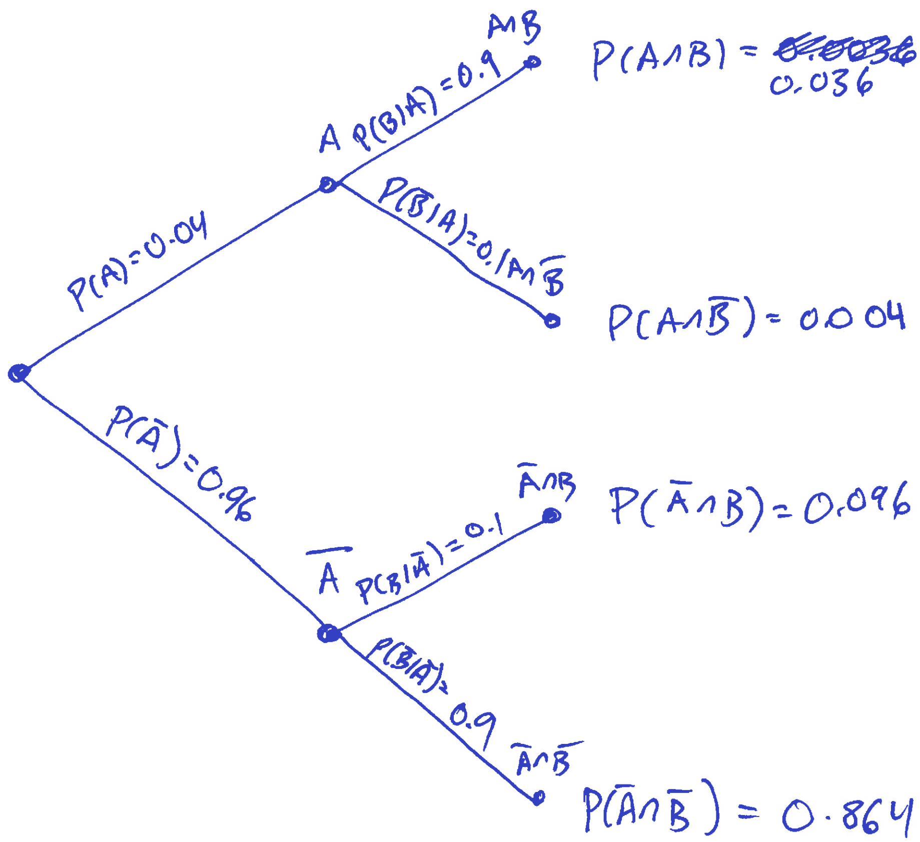
$$\boxed{= 0.9861}$$

5. (8) In a poker hand, John has a very strong hand and bets 5 dollars. The probability that Mary has a better hand is 0.04. If Mary had a better hand she would raise with probability 0.9, but with a poorer hand she would only raise with probability 0.1. If Mary raises, what is the probability that she has a better hand than John does?

Tree Diagrams are perfect for this.

A = Mary has a better hand

B = Mary raises



$$\begin{array}{r} {}^{5}96 \\ 9 \\ \hline 864 \end{array}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A \cap B) + P(\bar{A} \cap B)} = \frac{0.036}{0.036 + 0.096} = \frac{36}{132} = \frac{18}{66}$$

$$= \frac{9}{33} = \boxed{\frac{3}{11}}$$

6. (6)

(a) A fair ten-sided die labeled with the numbers 1 through 10 is rolled 200 times. Let X be the random variable for the number of times a prime number is rolled ($2, 3, 5, 7$ are prime, $1, 4, 6, 8, 9, 10$ are not). Find the probability distribution function for X and find $\mathbb{E}[X]$.

This is a Binomial Distribution with $n = 200$ and $p = \frac{4}{10} = \frac{2}{5}$.

So, $\boxed{P(X=k) = \binom{200}{k}\left(\frac{2}{5}\right)^k\left(\frac{3}{5}\right)^{200-k}}$.

$\mathbb{E}[X] = np = 200 \cdot \frac{2}{5} = \boxed{80.}$

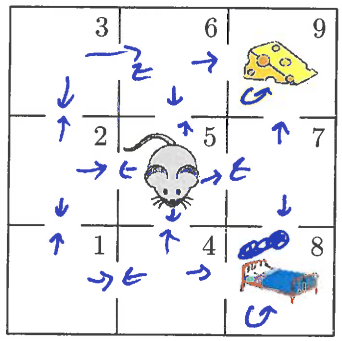
(b) A fair ten-sided die labeled with the numbers 1 through 10 is rolled until a prime number comes up. Let Y be the random variable for the total number of rolls. Find the probability distribution function for Y and find $\mathbb{E}[Y]$.

This is a geometric distribution with $p = \frac{2}{5}$.

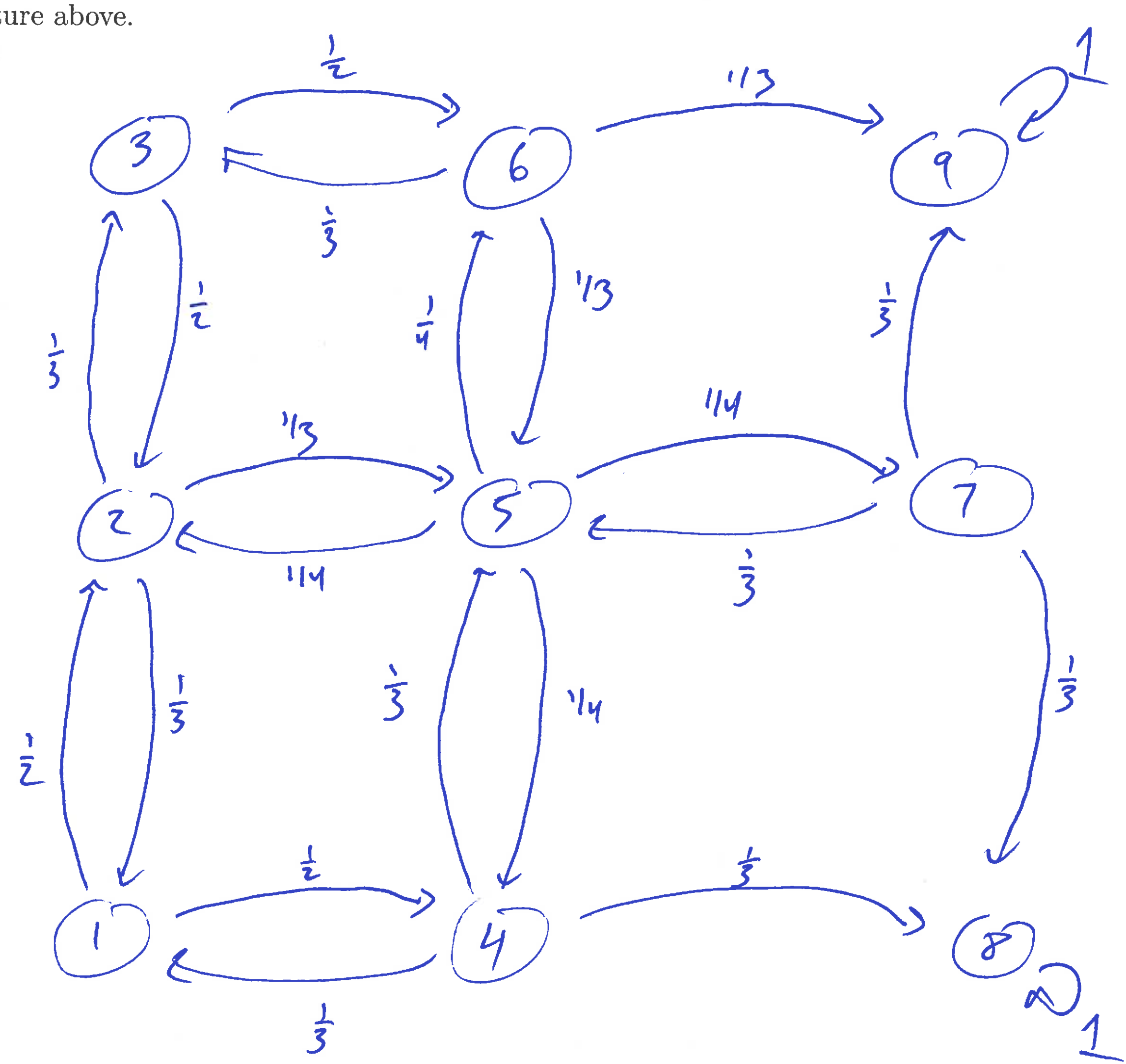
$\boxed{P(Y=k) = \left(\frac{3}{5}\right)^{k-1} \cdot \left(\frac{2}{5}\right)}$

$\mathbb{E}[Y] = \frac{1}{p} = \frac{5}{2} = \boxed{2.5.}$

7. (16) A mouse is put into the middle square of a 3×3 maze as shown below. At each time step, it moves from the room it is in to one of the adjacent rooms with equal probability. If the mouse reaches the room with the cheese or the room with a bed, it stays there forever.



(a) Draw the Markov chain for this process. Label the states using the numbers shown in the picture above.



(b) Write down the transition matrix $\boldsymbol{P}$ for the Markov chain. Use the order of the states given by the numbers in the picture (i.e., row 1 / column 1 represents room 1, etc.) Put a box around the part of the matrix that we call $\boldsymbol{Q}$. Put another box around the part of the matrix that we call $\boldsymbol{R}$. Be sure to label both boxes.

Q R

$$
\begin{array}{c|ccccccc|cc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
2 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
3 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
4 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\
5 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\
6 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} \\
7 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\
\hline
8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

(c) Is this Markov chain absorbing? Is it regular? Is it ergodic? Make sure to *fully* justify your assertions.

A MC is absorbing if an absorbing state can be reached from every state.

Any starting state can reach both 8 and 9, which are absorbing because they transition to themselves w/ prob. 1. So, this MC is absorbing.

Absorbing chains cannot be regular or ergodic (can never reach 1 from 8, for example.) So no.

(d) Let $\boldsymbol{Q}$ be as in part (b). You may now assume that

$$(\boldsymbol{I}-\boldsymbol{Q})^{-1} = \begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{pmatrix} \frac{21}{10} & \frac{15}{8} & \frac{9}{10} & \frac{57}{40} & \frac{3}{2} & \frac{33}{40} & \frac{3}{8} \\ \frac{5}{4} & \frac{43}{16} & \frac{5}{4} & \frac{17}{16} & \frac{7}{4} & \frac{17}{16} & \frac{7}{16} \\ \frac{9}{10} & \frac{15}{8} & \frac{21}{10} & \frac{33}{40} & \frac{3}{2} & \frac{57}{40} & \frac{3}{8} \\ \frac{19}{20} & \frac{17}{16} & \frac{11}{20} & \frac{143}{80} & \frac{5}{4} & \frac{47}{80} & \frac{5}{16} \\ \frac{3}{4} & \frac{21}{16} & \frac{3}{4} & \frac{15}{16} & \frac{9}{4} & \frac{15}{16} & \frac{9}{16} \\ \frac{11}{20} & \frac{17}{16} & \frac{19}{20} & \frac{47}{80} & \frac{5}{4} & \frac{143}{80} & \frac{5}{16} \\ \frac{1}{4} & \frac{7}{16} & \frac{1}{4} & \frac{5}{16} & \frac{3}{4} & \frac{5}{16} & \frac{19}{16} \end{pmatrix} \end{array}$$

What is the expected number of steps until the mouse reaches either the cheese or the bed? (You don't have to simplify your answer; you may leave it as a sum of fractions.)

Let $N = (I-Q)^{-1}$. Then $t = N\begin{pmatrix}\vdots\end{pmatrix}$ is the column vector of expected time to absorption depending on start state.

So,

answer = sum of row 5

$$= \boxed{\frac{3}{4} + \frac{21}{16} + \frac{3}{4} + \frac{15}{16} + \frac{9}{4} + \frac{15}{16} + \frac{9}{16}}$$

(e) You may now assume that

$$
NR = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c} \begin{array}{cc} 8 & 9 \end{array} \\
\begin{pmatrix}
\frac{3}{5} & \frac{2}{5} \\
\frac{1}{2} & \frac{1}{2} \\
\frac{2}{5} & \frac{3}{5} \\
\frac{7}{10} & \frac{3}{10} \\
\frac{1}{2} & \frac{1}{2} \\
\frac{3}{10} & \frac{7}{10} \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix} \end{array}.
$$

What is the probability that the mouse ends up in the cheese room?

cheese room = 9

entry in row 5 col 9

$= \frac{1}{2}$ (makes intuitive sense)

(f) What is the probability that the mouse ends up in the cheese room given that it is currently in room 3?

Since a MC is memoryless, being currently in room 3 is equivalent to starting in room 3. So,

entry in row 3 col 9

$= \frac{3}{5}$

(g) What is the probability that the mouse never visits either the cheese room or the bed room?

With probability 1, every process on an absorbing MC gets absorbed. So the prob. ~~of~~ that it never gets absorbed is $\boxed{0.}$

(h) Lastly, suppose that the cheese and bed are removed, so that the mouse moves around among all the rooms with no absorbing states. Prove that the resulting chain is ergodic but not regular.

Ergodic: It is clear that regardless of where the mouse starts, it can eventually (in at most 4 steps) reach any state. This is the definition of ergodic.

Not regular: Call the 4 corners and the center the RED squares, and the 4 edges the BLUE squares.

~~Then~~ After an even # of steps, the mouse can only be in a room the same color as where it started. After an odd #, only in a different color. Hence no power of the transition matrix can have all pos. entries.

8. (8) Let X and Y be independent random variables with probability density functions

$$f_X(x) = \begin{cases} \frac{1}{2}, & x \in [0,2] \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad f_Y(y) = \begin{cases} \lambda e^{-\lambda y}, & y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Let $Z = X + Y$. Find the probability density function $f_Z(z)$ of Z.

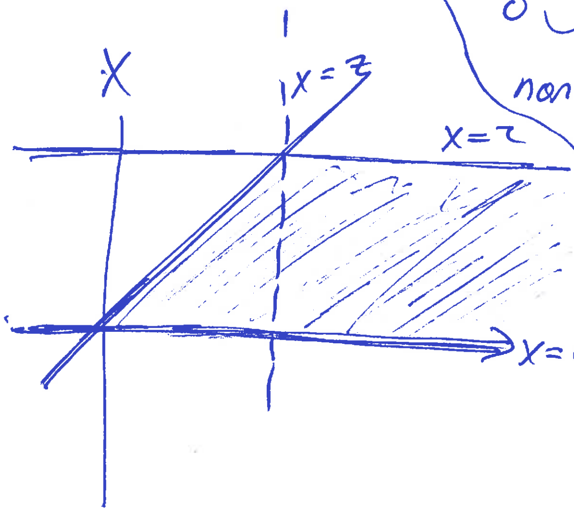
$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$$

$$= \frac{1}{2}\int_0^z \underbrace{f_Y(z-x)}\,dx$$

(because $f_X(x)$ is nonzero when $x \in [0,2]$ only, and is $= \frac{1}{2}$ there).

non zero when $z - x \geq 0$

$\Rightarrow x \leq z$



Two integrals.

$\boxed{\text{When } z \in [0,2]:}$

$$f_Z(z) = \frac{1}{2}\int_{x=0}^{x=z} \lambda e^{-\lambda(z-x)}\,dx$$

$$= \frac{1}{2}\lambda e^{-\lambda z}\int_{x=0}^{x=z} e^{\lambda x}\,dx$$

$$= \frac{1}{2}\lambda e^{-\lambda z}\left[\frac{1}{\lambda}e^{\lambda x}\right]_{x=0}^{x=z}$$

$$= \frac{1}{2}e^{-\lambda z}\left(e^{\lambda z} - 1\right)$$

$$= \frac{1}{2}\left(1 - e^{-\lambda z}\right)$$

When $z \in [2, \infty]$:

$$f_Z(z) = \frac{1}{2}\int_{x=0}^{x=2} \lambda e^{-\lambda(z-x)}\,dx$$ [same integral different bounds]

$$= \frac{1}{2}e^{-\lambda z}\left[e^{\lambda x}\right]_{x=0}^{x=2}$$

$$= \frac{1}{2}e^{-\lambda z}\left(e^{2\lambda} - 1\right)$$

$$= \frac{1}{2}e^{-\lambda(z-2)} - \frac{1}{2}e^{-\lambda z}$$

So, $$f_Z(z) = \begin{cases} \frac{1}{2}\left(1 - e^{-\lambda z}\right), & 0 \leq z \leq 2 \\ \frac{1}{2}\left(e^{-\lambda(z-2)} - e^{-\lambda z}\right), & 2 \leq z \leq \infty \\ 0, & \text{otherwise} \end{cases}$$

9. (0) **Bonus:** *(3 points)* Tell me a little bit about how you prepared for this class (what were your study techniques, did you cram or spread out the work, etc.) What worked for you and what didn't? This will help me give advice to future classes.

Standard Normal Probabilities

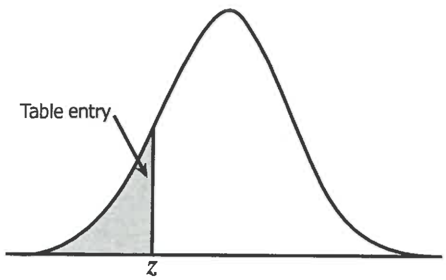


Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Standard Normal Probabilities

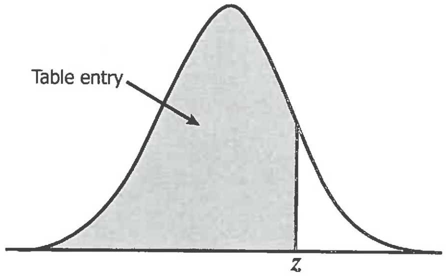


Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998