Math 20 Spring 2013

Discrete Probability

Final Exam

Friday May 31, 8:00–11:00 AM

Your name (please print): ____

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You must justify your answers to receive full credit. Partial credit will be awarded for serious progress toward a solution, but not for just writing something down. In addition, partial credit will be awarded for interesting ideas, even if they don't end up working. The instructor also reserves the right to award more than full credit for especially interesting (correct) solutions.

A table of normal probabilities is included on the last page.

The Honor Principle requires that you neither give nor receive any aid on this exam.

For grader use only:

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
Total	120	

1. Let X be a random variable with mean 0 and variance 1. Find the smallest number n so that you can guarantee that $P(|X| \geq n) \leq 1/25$.

   By Chebyshev's inequality, we have

$$P(|X| \geq n) \leq \frac{V(X)}{n^2} = \frac{1}{n^2},$$

   so if $n \geq 5$, then we are guaranteed that $P(|X| \geq n) \leq \frac{1}{25}$. Hence $n = 5$ is smallest such number.

2. Let X and Y be independent Poisson random variables with means λ and μ, respectively. Show that $X+Y$ is also Poisson with mean $\lambda+\mu$.

Solution 1: We have

$$
\begin{aligned}
P(X+Y=n) &= \sum_{k=0}^{n} P(X=k)P(Y=n-k) \\
&= \sum_{k=0}^{n} e^{-\lambda}\frac{\lambda^k}{k!}e^{-\mu}\frac{\mu^{n-k}}{(n-k)!} \\
&= e^{-(\lambda+\mu)}\sum_{k=0}^{n}\frac{\lambda^k\mu^{n-k}}{k!(n-k)!} \\
&= \frac{e^{-(\lambda+\mu)}}{n!}\sum_{k=0}^{n}\binom{n}{k}\lambda^k\mu^{n-k} \\
&= \frac{e^{-(\lambda+\mu)}}{n!}(\lambda+\mu)^n,
\end{aligned}
$$

as desired.

Solution 2: We show that the generating function of the sum of X and Y is the same as the generating function of a Poisson random variable with mean $\lambda+\mu$. In order to do this, we compute the generating function for a Poisson distribution with mean λ: it is

$$
\begin{aligned}
P_\lambda(x) &= \sum_{n=0}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}x^n \\
&= e^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda x)^n}{n!} \\
&= e^{-\lambda}e^{\lambda x} \\
&= e^{\lambda(x-1)}.
\end{aligned}
$$

Similarly, the generating functions for Poisson distributions with means μ and $\lambda+\mu$ are

$$P_\mu(x) = e^{\mu(x-1)}, \qquad P_{\lambda+\mu}(x) = e^{(\lambda+\mu)(x-1)}.$$

Now, the generating function for $X+Y$ is the product of the generating functions for X and Y, so it is

$$P_\lambda(x)P_\mu(x) = e^{\lambda(x-1)}e^{\mu(x-1)} = e^{(\lambda+\mu)(x-1)} = P_{\lambda+\mu}(x),$$

as desired.

3. Find (to as good of an approximation as you can) the number n so that, when a fair coin is flipped 10000 times, the probability of observing between 4930 and n heads is 1/2.

   The mean is $\mu = 5000$ and the standard deviation is $\sqrt{np(1-p)} = \sqrt{10000/4} = 50$. The standardized value for 4930 is

$$\frac{4930 - 5000}{50} = -1.4.$$

   The probability of getting a standardized value between -1.4 and 0 is 0.4192, so we need to find out what standardized value gives us the remaining $1/2 - 0.4192 = 0.0808$. This corresponds to a standardized value between 0.2 and 0.21; let us use 0.2 for convenience. The number whose standardized value is 0.2 satisfies

$$\frac{X - 5000}{50} = 0.2,$$

   or $X = 5010$.

4. A casino offers people the chance to play the following game: flip two fair coins. If both come up heads, the gambler wins \$1. If both come up tails, the gambler wins \$3. If one is heads and one is tails, the gambler gets nothing. The game costs \$1.25 to play. Your friend, who has not taken a probability course and thus doesn't know any better, goes to this casino and plays the game 600 times. Estimate the probability that your friend loses between \$132 and \$195 over the course of the 600 games.

Let us first work out the expected value and variance for the result X of one game. We have

$$P(X = -5/4) = \frac{1}{2}, \qquad P(X = -1/4) = \frac{1}{4}, \qquad P(X = 7/4) = \frac{1}{4},$$

so

$$\mathbb{E}(X) = \frac{1}{2}\left(-\frac{5}{4}\right) + \frac{1}{4}\left(-\frac{1}{4}\right) + \frac{1}{4}\left(\frac{7}{4}\right) = -\frac{1}{4}.$$

To work out the variance, we use the formula $V(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$, so we must compute $\mathbb{E}(X^2)$. We have

$$\mathbb{E}(X^2) = \frac{1}{2}\left(\frac{25}{16}\right) + \frac{1}{4}\left(\frac{1}{16}\right) + \frac{1}{4}\left(\frac{49}{16}\right) = \frac{25}{16}.$$

Hence

$$V(X) = \frac{25}{16} - \frac{1}{16} = \frac{3}{2}.$$

So the standard deviation is $\sigma = \sqrt{3/2}$. The mean over 600 games is $600\mu = -150$, or an expected loss of 150; the standard deviation over 600 games is $\sigma\sqrt{600} = 30$. The standardized value for 132 is

$$132^* = \frac{132 - 150}{30} = -0.6,$$

and the standardized value for 195 is

$$195^* = \frac{195 - 150}{30} = 1.5.$$

Now, the probability that we see a standardized score between -0.6 and 1.5 is $0.2257 + 0.4332 = 0.6589$.

5. Consider the Markov chain with states labelled 0,1,2,3,4 (in that order), and transition probabilities given by

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/3 & 1/6 & 0 \\ 0 & 1/6 & 1/2 & 1/6 & 1/6 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Verify that

$$\mathbf{N} = \frac{1}{25}\begin{pmatrix} 30 & 30 & 15 \\ 12 & 72 & 21 \\ 6 & 36 & 48 \end{pmatrix}$$

is the fundamental matrix for this chain.

We have

$$\mathbf{I} - \mathbf{Q} = \begin{pmatrix} 1 & -1/3 & -1/6 \\ -1/6 & 1/2 & -1/6 \\ 0 & -1/3 & 2/3 \end{pmatrix};$$

when we multiply this by $\mathbf{N}$, we get $\mathbf{I}$.

(b) If the chain starts in state 1, what is the probability that it gets absorbed in state 4?

The answer is the 1–4 entry of the matrix $\mathbf{NR}$, which is $\frac{10}{25} = \frac{2}{5}$.

(c) What is the expected number of times we visit state 2 if we start at state 1?

This is the 1–2 entry of $\mathbf{N}$, which is $\frac{30}{25} = \frac{6}{5}$.

6. (a) Write down the transition matrix for a Markov chain which is neither absorbing nor ergodic.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(b) Write down the transition matrix for a Markov chain which is ergodic but not regular.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(c) Write down the transition matrix for a regular Markov chain in which it is possible to get from any state to any state in 3 steps, but it is not possible to get from any state to any state in fewer than 3 steps.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

7. A rook does a random walk on a (standard 8×8) chessboard, starting from the bottom right corner. (On its move, a rook can move to any square on the same horizontal row or vertical column as it started, except for its original square.) What is the expected number of steps it takes to return to the bottom right corner? Explain why. (This problem would be much harder for other chess pieces.)

   The chessboard is completely symmetric as far as rook moves go; hence the fixed vector is the uniform vector, all of whose entries are $1/64$. Hence the mean recurrence time is $1/w_i = 64$.

8. Suppose that X is a random variable that takes nonnegative integer values, and let $p_n = P(X = n)$. Suppose the generating function of X is

$$A(x) = \sum_{n=0}^{\infty} p_n x^n = \frac{1}{4 - 3x}.$$

Find $\mathbb{E}(X)$ and $V(X)$.

We have

$$A(x) = \sum_{n=0}^{\infty} p_n x^n.$$

Then

$$\mathbb{E}(X) = \sum_{n=0}^{\infty} n p_n.$$

We can obtain this by differentiation of A: we have

$$A'(x) = \sum_{n=0}^{\infty} n p_n x^{n-1},$$

so

$$\mathbb{E}(X) = A'(1).$$

Computing this, we find that

$$A'(x) = \frac{3}{(4 - 3x)^2},$$

so $\mathbb{E}(X) = A'(1) = 3$.

In order to compute the variance, we need to compute $\mathbb{E}(X^2)$, or

$$\sum_{n=0}^{\infty} n^2 p_n.$$

To do this, we differentiate A twice, but we can be a little bit clever about how we do it: we have

$$A'(x) = \sum_{n=0}^{\infty} n p_n x^{n-1},$$

and if we differentiate again, we get coefficients of $n(n-1)$ instead of n^2, so we first multiply by x to get

$$xA'(x) = \sum_{n=0}^{\infty} n p_n x^n.$$

Now, differentiating again, we obtain

$$\frac{d}{dx}(xA'(x)) = A'(x) + xA''(x) = \sum_{n=0}^{\infty} n^2 p_n x^{n-1}.$$

Hence

$$EE(X^2) = A'(1) + A''(1).$$

So, we compute $A''(x)$, and we find that

$$A''(x) = \frac{18}{(4-3x)^3}.$$

Hence

$$V(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = A'(1) + A''(1) - A'(1)^2 = 3 + 18 - 9 = 12.$$

(Another possible method is to note that X is almost a geometric random variable with success probability $1/4$; in fact, it is one less than a geometric random variable. From there we can either recall or rederive the statistics for a geometric random variable and hence X.)

9. You have ten envelopes, labelled 1 through 10, and 20 letters, labelled 1 through 10, and there are exactly two letters with each label. You randomly put two letters in each envelope. What is the expected number of envelopes that contain at least one of the two letters with the same number as the envelope?

   As usual, we use linearity of expectation: let X_i be the random variable that takes on the value 1 if at least one letter labelled i is in the i^{th} envelope, and 0 otherwise. Then

$$\mathbb{E}(X_i) = P(X_i = 1) = 1 - \frac{18}{20} \times \frac{17}{19} = \frac{37}{190}.$$

   Then the expected value for the number of envelopes containing at least one correct letter is

$$\mathbb{E}(X_1 + X_2 + \cdots + X_{10}) = 10 \times \frac{37}{190} = \frac{37}{19}.$$

10. You have 100 coins, of which 95 are fair, four land on heads with probability 4/5, and one lands on heads with probability 1. Suppose you pick a coin at random, flip it 10 times, and get 10 heads. What is the probability that the coin you chose lands on heads with probability 4/5?

By Bayes's Theorem, we have

$$\begin{aligned} P(4/5 \mid 10) &= \frac{P(10 \mid 4/5) \times P(4/5)}{P(10)} \\ &= \frac{(4/5)^{10} \times \frac{1}{25}}{(1/2)^{10} \times 19/20 + (4/5)^{10} \times 1/25 + 1/100}. \end{aligned}$$

11. Suppose that n is a positive integer, and k is a positive integer between 0 and n. Find a closed form (i.e., something without a Σ or $\cdots$) for

$$\sum_{j=0}^{k}(-1)^j\binom{n}{j}.$$

Give a proof that your answer is correct.

We use Pascal's identity to write

$$\binom{n}{j}=\binom{n-1}{j-1}+\binom{n-1}{j}.$$

We can replace the $\binom{n}{j}$ in the sum directly, but it is easier to see what is going on if we unfold it a bit: we have

$$\begin{aligned}\sum_{j=0}^{k}(-1)^j\binom{n}{j}&=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^k\binom{n}{k}\\&=\binom{n-1}{0}-\binom{n-1}{0}-\binom{n-1}{1}+\binom{n-1}{1}+\binom{n-1}{2}\\&\quad-\cdots+(-1)^k\binom{n-1}{k-1}+(-1)^k\binom{n-1}{k}\\&=(-1)^k\binom{n-1}{k},\end{aligned}$$

since all the other terms cancel.

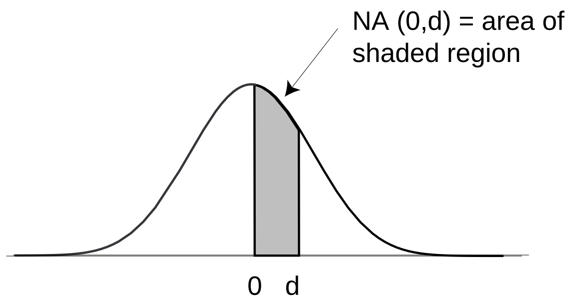
12. In order to ensure that you don't suffer from math withdrawal syndrome now that your Math 20 course has finished, you decide that, starting tomorrow, you will read a chapter of some math book each morning. Each day you choose randomly among *Visual Complex Analysis* by Needham, *Introduction to Analytic Number Theory* by Apostol, and *A Probability Path* by Resnick, each with probability 1/3. What is the probability that on the 11^{th} day from now (so the 11^{th} day of reading), you read the fourth chapter of *Visual Complex Analysis*?

(Of course, the point of this question was to recommend books for you to read over the summer.) If you read chapter 4 of *Visual Complex Analysis* on day 11, then you must have read three chapters in the first 10 days, with probability $\binom{10}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^7$, and then you must choose *Visual Complex Analysis* on day 11, with probability $\frac{1}{3}$. Hence the final answer is

$$\binom{10}{3}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^7.$$

Appendix A

Normal distribution table



	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000