MATH 20, WORKSHEET 7

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1. Have you taken linear algebra in the past?

- **No**. Find someone that has taken linear algebra and work with them through the rest of the class.

- **Yes**. You are responsible to guide someone through rest of the worksheet.

2. Matrices

A matrix is a rectangular arrangement of numbers into rows and columns. (You can think of this as a generalization of a vector). For example, matrix A has two rows and three columns.

$$A = \begin{pmatrix} -2 & 5 & 6 \\ 5 & 2 & 7 \end{pmatrix}.$$

The dimensions of a matrix tells its size: the number of rows and columns of the matrix, in that order. Since matrix A has two rows and three and columns we write its dimensions as 2×3, pronounced "two by three". In constrast, matrix B has three rows and two columns, so it is a 3×2 matrix.

$$B = \begin{pmatrix} -8 & -4 \\ 23 & 12 \\ 18 & 10 \end{pmatrix}.$$

When working with matrix dimensions, remember rows $\times$ columns.

3. Matrix elements

A **matrix element** is simply a matrix entry. Each element in a matrix is identified by naming the row and column in which it appears. For example, consider the matrix G:

$$G = \begin{pmatrix} 4 & 14 & -7 \\ 18 & 5 & 13 \\ -20 & 4 & 22 \end{pmatrix}$$

The element $G_{2,1} = 18$ is the first entry on the second row.

In general, the element in row i and column j of matrix A is denoted by $A_{i,j}$.

- Test your understanding with some examples.

- What is $B_{1,2}$ and $B_{1,1}$?

4. Addition

Just like vectors, if two matrices have the same dimensions, you can add them, by adding their entries. Using the notation of matrix elements, the entries of the sum are given by

$$(A+B)_{i,j} = A_{i,j} + B_{i,j}$$

- Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \end{pmatrix}$

  Compute $A+B$, $A-B$
- Add $A = \begin{pmatrix} 4 & 14 & -7 \\ 18 & 5 & 13 \\ -20 & 4 & 22 \end{pmatrix}$ and $B = \begin{pmatrix} -8 & -4 & 1 \\ 23 & 12 & 2 \\ 18 & 10 & 4 \end{pmatrix}$.

5. Multiplication by a scalar

Also as vectors, we can multiply a matrix by a scalar α, by multiplying each entry.

$$(\alpha A)_{i,j} = \alpha A_{i,j}$$

- Check that $2A = A + A$.

6. Matrix multiplication

Matrices can be multiplied **only** if the first matrix has as many columns as the second matrix has rows.

- A is a $m \times n$ matrix
- B is a $n \times k$ matrix
- then $A \cdot B$ is a $m \times k$ matrix

The element in the ith row and jth column of $A \cdot B$ is

$$(A \cdot B)_{ij} = \sum_{\ell=1}^{k} A_{i\ell} B_{\ell j}$$

In other words, we have to transverse the ith row of A and the jth column of B and sum the product of the corresponding entries.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 1\cdot 1 + 2\cdot 0 + 1\cdot 2 & 1\cdot 0 + 2\cdot 1 + 1\cdot 3 \\ 3\cdot 1 + 2\cdot 0 + 3\cdot 2 & 3\cdot 0 + 2\cdot 1 + 3\cdot 3 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 9 & 11 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 11 & 10 & 11 \end{pmatrix} \neq A \cdot B$$

Matrix multiplication is not commutative! (not even of the same shape)

- Check the example above.
- Multiply $A = \begin{pmatrix} 4 & 14 \\ 18 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & 1 \\ 3 & 2 \end{pmatrix}$.
- Let $A = \begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -4 \\ -1 & 5 \end{pmatrix}$. Compute $A \cdot B$ and $B \cdot A$.

It is **associative** and **distributive** (if all the operations are defined)

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$
$$A \cdot (B + C) = A \cdot B + A \cdot C$$
$$(B + C) \cdot A = B \cdot A + C \cdot A$$

The **Identity** matrix is a $n \times n$ matrix, with 1s on the diagonal and zeros everywhere else.

$$I_{n\times n} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

If A is a $m \times n$ matrix

$$A \cdot I_{n\times n} = A = I_{m\times m} \cdot A$$

- Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \dfrac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Verify that $A \cdot B = B \cdot A = I_{2\times 2}$. In this case, we say that B is the inverse of A, and we denote B by A^{-1}.

7. Other problems

- Let P be a $r \times r$ matrix, where we denote i, j-entry by $p_{i,j}$. Show that the i, j-entry of P^2, $\left(P^2\right)_{i,j}$, is given by $\displaystyle\sum_{k=1}^{r} p_{i,k} p_{k,j}$
- Show that if each row of P adds up to 1, i.e., $\displaystyle\sum_{j=1}^{r} p_{i,j} = 1$ for all i, then the same holds for P^2.
- Repeat the above questions for P^3.
- Can you generalize the above formulas?