MATH 20, WORKSHEET 4

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Due Friday October 2nd

(1) Let X be a discrete random variable that takes only positive integer values. Our normal formula for the expected value of X says

$$E[X] = \sum_{k=1}^{+\infty} kP(X = k).$$

Prove the following alternate formula:

$$E[X] = \sum_{k=1}^{+\infty} P(X \geq k).$$

Answer: Since k is always an integer, we can use the fact that

$$k \cdot P(X = k) = \underbrace{P(X = k) + P(X = k) + \cdots + P(X = k)}_{k \text{ times}}.$$

Note also that since X takes only values in the positive integer we may write

$$P(X \geq k) = P(X = k) + P(X = k + 1) + \cdots \,. \tag{0.1}$$

Suppose that we take the sum

$$\sum_{k=1}^{\infty} P(X \geq k)$$

and use equation (1) to expand. We get

$$\sum_{k=1}^{\infty} (P(X = k) + P(X = k + 1) + P(X = k + 2) + \cdots).$$

How often is a particular term $P(X = N)$ counted in this sum? It appears once in $P(X \geq 1)$, once in $P(X \geq 2)$, etc, all the way up to $P(X \geq N)$, and then it doesn't appear in the rest of the terms. Thus, $P(X = N)$ is counted N times—once in each of the first N terms. Therefore,

$$\sum_{k=1}^{\infty} P(X \geq k) = \sum_{k=1}^{\infty} kP(X = k) = E[X]$$

and the identity is proved.

(2) Compute the expected value of the geometric distribution with parameter p.

Answer: X only takes values in the positive integers. Using the above formula

$$\begin{aligned} E[X] &= \sum_{k=1}^{+\infty} P(X \geq k) \\ &= \sum_{k=1}^{+\infty} (1-p)^{k-1} \\ &= \sum_{l=0}^{+\infty} (1-p)^{l} \\ &= \frac{1}{1-(1-p)} = \frac{1}{p} \qquad \text{(Assuming } p > 0\text{)} \end{aligned}$$

For $p = 0$,

(3) Recall that if $P(A) > 0$, then $X|A$ is a random variable where

$$m_{X|A}(x) = P(X = x|A) = \begin{cases} \dfrac{m_X(x)}{P(A)}, & \text{if } x \in A \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$E[X|A] = \sum_{x \in \Omega} x \cdot P(X = x|A).$$

(a) Assume that $P(\text{not } A) > 0$. Write $E[X]$ in terms of $E[X|A]$ and $E[X|\text{not } A]$. (Hint: see the formula below)

Answer: The formula desired is

$$E[X] = P(A)E[X|A] + P(\text{not } A)E[X|\text{not } A].$$

This is part (b) below with $k = 2$, $F_1 = A$ and $F_2 =$ not A.

(b) Assume that $F_1, F_2, \ldots, F_k$ are events with positive probability, pair wise disjoints and $\Omega = F_1 \cup F_2 \cup \cdots \cup F_k$. Show that

$$E[X] = \sum_{j=1}^{k} E[X|F_j]P(F_j).$$

Answer: See Theorem 6.5 in Grinstead and Snell's Introduction to Probability.