MATH 20, WORKSHEET 3

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(1) You are given two boxes and an even number of balls. Half of the balls are white and half are black. You are asked to distribute the balls in the boxes with no restriction placed on the number of either type in an box. How should you distribute the balls in the boxes to maximize the probability of obtaining a white ball if an box is chosen at random and a ball drawn out at random?
Hints:
- Start by showing that you can do better than having the same number of white and black balls in each box.
- What are the odds if you put all the black balls in one box and all the white in the other box? If you move one black (or white) ball from one box to the other, what happens to your odds?

Answer:
The best way to distribute the balls to maximize the probability of obtaining a white ball if an box is chosen at random and a ball drawn out at random is to place one white ball in one box and all the others balls in the other box.

This gives a probability of nearly $3/4$, in particular greater than $1/2$, for obtaining a white ball which is what you would have with an equal number of balls in each box. Therefore, the best choice must have more white balls in one box than the other.

In the box with more white balls than black, the best we can do is to have probability 1, i.e., all white balls. In the box with less white balls than black, the best we can do is to have probability closer to $1/2$ has possible, i.e., to have one less white ball than black and then to have as many white balls as possible.

Our solution is thus best for the box with more white balls than black and also for the box with more black balls than white. Therefore our solution is the best we can do.

(2) A deck of playing cards can be described as a Cartesian product

$$\text{Deck} = \text{Suit} \times \text{Rank}\ ,$$

where Suit $= \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$ and Rank $= \{2, 3, \ldots, 10, \text{J}, \text{Q}, \text{K}, \text{A}\}$. This just means that every card may be thought of as an ordered pair like $(\diamondsuit, 2)$. By a **suit event** we mean any event A contained in Deck which is described in terms of Suit alone. For instance, if A is "the suit is red," then

$$A = \{\diamondsuit, \heartsuit\} \times \text{Rank}\ ,$$

so that A consists of all cards of the form $(\diamondsuit, r)$ or $(\heartsuit, r)$ where r is any rank. Similarly, a **rank event** is any event described in terms of rank alone.

(a) Show that if A is any suit event and B any rank event, then A and B are independent. (We can express this briefly by saying that suit and rank are independent.)

Answer:
Since all the cards are equally likely to be drawn, for a event $E \subset$ Deck, we have

$$P(E) = \frac{\#E}{52},$$

in other words, our distribution function is $m(s, r) = 1/52$.
A suit event A can always be written as

$$A = \{(s, r) \in \text{Deck} : s \in S_A \subset \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}\},$$

and therefore

$$P(A) = \frac{\#A}{52} = \frac{13 \cdot \#S_A}{52} = \frac{\#S_A}{4}.$$

Similarly, a rank event B can always be written as

$$B = \{(s,r) \in \text{Deck} : r \in R_B \subset \{2,3,\ldots,10,\text{J},\text{Q},\text{K},\text{A}\}\},$$

and we have

$$P(B) = \frac{\#B}{52} = \frac{4 \cdot \#R_B}{52} = \frac{\#R_B}{13}.$$

For last,

$$A \cap B = \{(s,r) \in \text{Deck} : s \in S_A \text{ and } r \in R_B\},$$

and

$$P(A \cap B) = \frac{\#(A \cap B)}{52} = \frac{\#S_A \cdot \#R_B}{52} = \frac{\#S_A}{4} \cdot \frac{\#R_B}{13} = P(A) \cdot P(B).$$

Therefore, A and B are independent.

(b) Throw away the ace of spades. Show that now no nontrivial (i.e., neither empty nor the whole space) suit event A is independent of any nontrivial rank event B.
Hint: Here independence comes down to

$$c/51 = (a/51) \cdot (b/51) \ ,$$

where a, b, c are the respective sizes of A, B and $A \cap B$. It follows that 51 must divide ab, hence that 3 must divide one of a and b, and 17 the other. But the possible sizes for suit and rank events preclude this.

Answer:

Following the hint, and what we did in (a), let $a := \#A$, $b := \#B$, and $c := \#(A \cap B)$, we want to show that for $a, b \notin \{0, 51\}$ we have

$$\frac{c}{51} = P(A \cap B) \neq P(A) \cdot P(B) = \frac{a}{51} \cdot \frac{b}{51}.$$

(Once again, we are assuming that each card of this 51 deck is equally like to be drawn.)
We will prove this by contradiction, assume that equality holds given that $a, b \notin \{0, 51\}$. Multiplying both sides by 51, we get

$$3 \cdot 7 \cdot c = ab.$$

Therefore, 3 and 7 must divide $a \cdot b$ (regardless of the value of c). However,

$$b = \begin{cases} 4k+3 & \text{if } \spadesuit \in S_A; \\ 4k & \text{otherwise;} \end{cases}$$

for some $0 \leq k \leq 12$. Similarly, $a = 13l + 12$ or $13l$ for some $l \geq 0$. Explicitly,

$$a \in \{0, 12, 13, 25, 26, 38, 39, 51\}$$

$$b \in \{0, 3, 4, 7, 8, 11, 12, 15, 16, 19, 20, 23, 24, 27, 28, 31, 32, 35, 36, 39, 40, 43, 44, 47, 48, 51\}$$

and the only numbers divisible by 17 in the two sets above are 0 and 51, which contradcits $a, b \notin \{0, 51\}$.

(c) Show that the deck in (b) nevertheless does have pairs A, B of nontrivial independent events.
Hint: Find 2 events A and B of sizes 3 and 17, respectively, which intersect in a single point.

Answer:

Pick

$$A = \{(A, \clubsuit), (A, \diamondsuit), (A, \heartsuit)\}$$

$$B = \{(2, \clubsuit), (3, \clubsuit), \ldots (\text{K}, \clubsuit), (\text{A}, \clubsuit), (3, \spadesuit), (4, \spadesuit), (5, \spadesuit), (5, \spadesuit)\}.$$

We have, $A \cap B = \{(A, \clubsuit)\}$, $\#A = 3$ and $\#B = 17$, therefore

$$P(A \cap B) = \frac{1}{51} = \frac{3}{51}\frac{17}{51} = P(A) \cdot P(B)$$

.

(d) Add a joker to a full deck. Show that now there is no pair A, B of nontrivial independent events. Hint: See the hint in (b); 53 is prime.

Answer:

Follows the same argument as in (b), however now we have

$$53 \cdot c = a \cdot b.$$

Since 53 is prime, we have that 53 divides a or b. Without loss of generality assume that it divides a, since $a \leq 53$ we have $a \in \{0, 53\}$ which makes the A event trivial.