Problem

Show that

$$b(n,p,j) = \frac{p}{q}\left(\frac{n-j+1}{j}\right) b(n,p,j-1)\ ,$$

for $j \geq 1$. Use this fact to determine the value or values of j which give $b(n,p,j)$ its greatest value.

Problem

Show that the number of ways that one can put n different objects into three boxes with a in the first, b in the second, and c in the third is $n!/(a!\,b!\,c!)$.

Problem

Prove that the probability of exactly n heads in $2n$ tosses of a fair coin is given by the product of the odd numbers up to $2n - 1$ divided by the product of the even numbers up to $2n$.

Conditional Probability

10/04/2005

Example

Three candidates A, B, and C are running for office. We decided that A and B have an equal chance of winning and C is only $1/2$ as likely to win as A. Let A be the event "A wins," B that "B wins," and C that "C wins." Hence, we assigned probabilities $P(A) = 2/5$, $P(B) = 2/5$, and $P(C) = 1/5$.

Suppose that before the election is held, A drops out of the race. What are the values for $P(B|\ A)$ and $P(C|\ A)$?

Definition

Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_r\}$ be the original sample space with distribution function $m(\omega_j)$ assigned. Suppose we learn that the event E has occurred.

- If a sample point ω_j is not in E, we want $m(\omega_j|E) = 0$.

- For ω_k in E, we should have the same relative magnitudes that they had before we learned that E had occurred:

$$m(\omega_k|E) = cm(\omega_k).$$

But we must also have

$$\sum_E m(\omega_k|E) = c \sum_E m(\omega_k) = 1 \ .$$

Thus,

$$c = \frac{1}{\sum_E m(\omega_k)} = \frac{1}{P(E)} \ .$$

Definition 1. *The* conditional distribution *gven E is the distribution on Ω defined by*

$$m(\omega_k|E) = \frac{m(\omega_k)}{P(E)}$$

for ω_k in E, and $m(\omega_k|E) = 0$ for ω not in E.

Then, for a general event F,

$$P(F|E) = \sum_{F\cap E} m(\omega_k|E) = \sum_{F\cap E} \frac{m(\omega_k)}{P(E)} = \frac{P(F\cap E)}{P(E)} \ .$$

We call $P(F|E)$ the *conditional probability of F occurring given that E occurs.*

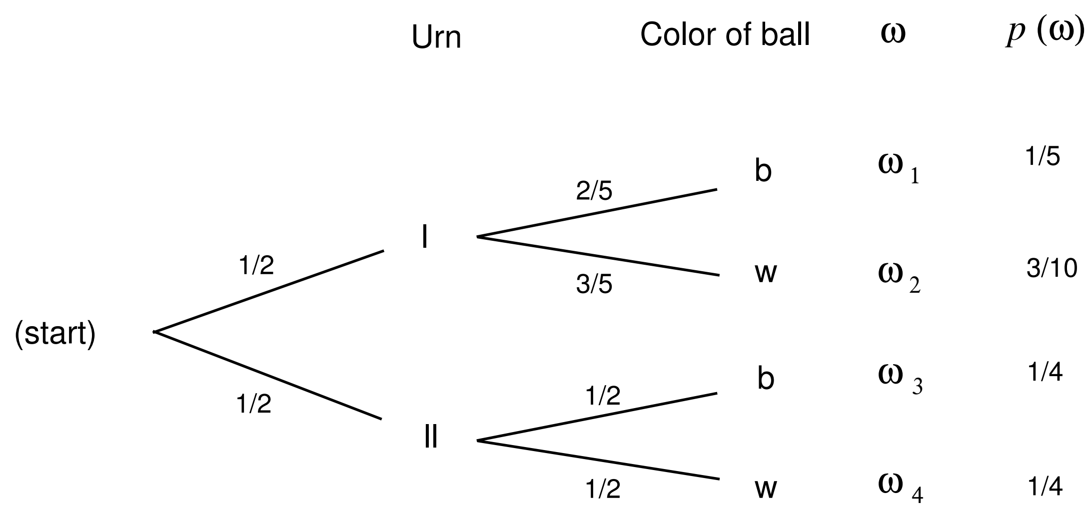
Example

Let us return to the example of rolling a die. Recall that F is the event $X = 6$, and E is the event $X > 4$. Note that $E \cap F$ is the event F. So, the above formula gives

$$
\begin{aligned}
P(F|E) &= \frac{P(F \cap E)}{P(E)} \\
&= \frac{1/6}{1/3} \\
&= \frac{1}{2}\,.
\end{aligned}
$$

Example

We have two urns, I and II. Urn I contains 2 black balls and 3 white balls. Urn II contains 1 black ball and 1 white ball. An urn is drawn at random and a ball is chosen at random from it. We can represent the sample space of this experiment as the paths through a tree.

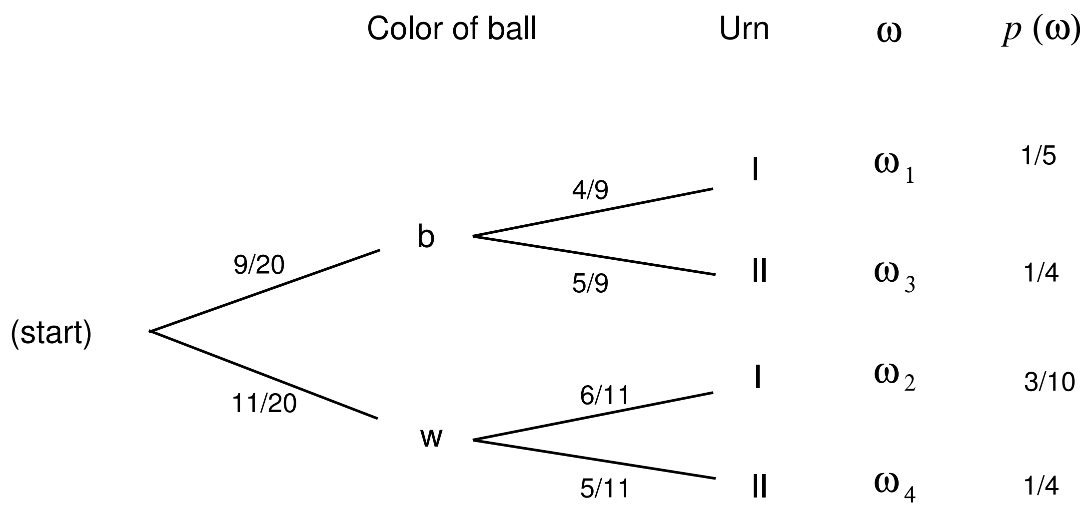

Urn
Color of ball
ω
p (ω)
(start)
1/2
1/2
I
II
2/5
3/5
1/2
1/2
b
w
b
w
ω 1
ω 2
ω 3
ω 4
1/5
3/10
1/4
1/4

- Let B be the event "a black ball is drawn," and I the event "urn I is chosen." Then the branch weight $2/5$, which is shown on one branch in the figure, can now be interpreted as the conditional probability $P(B|I)$.

- What is $P(I|B)$?

Bayes Probabilities

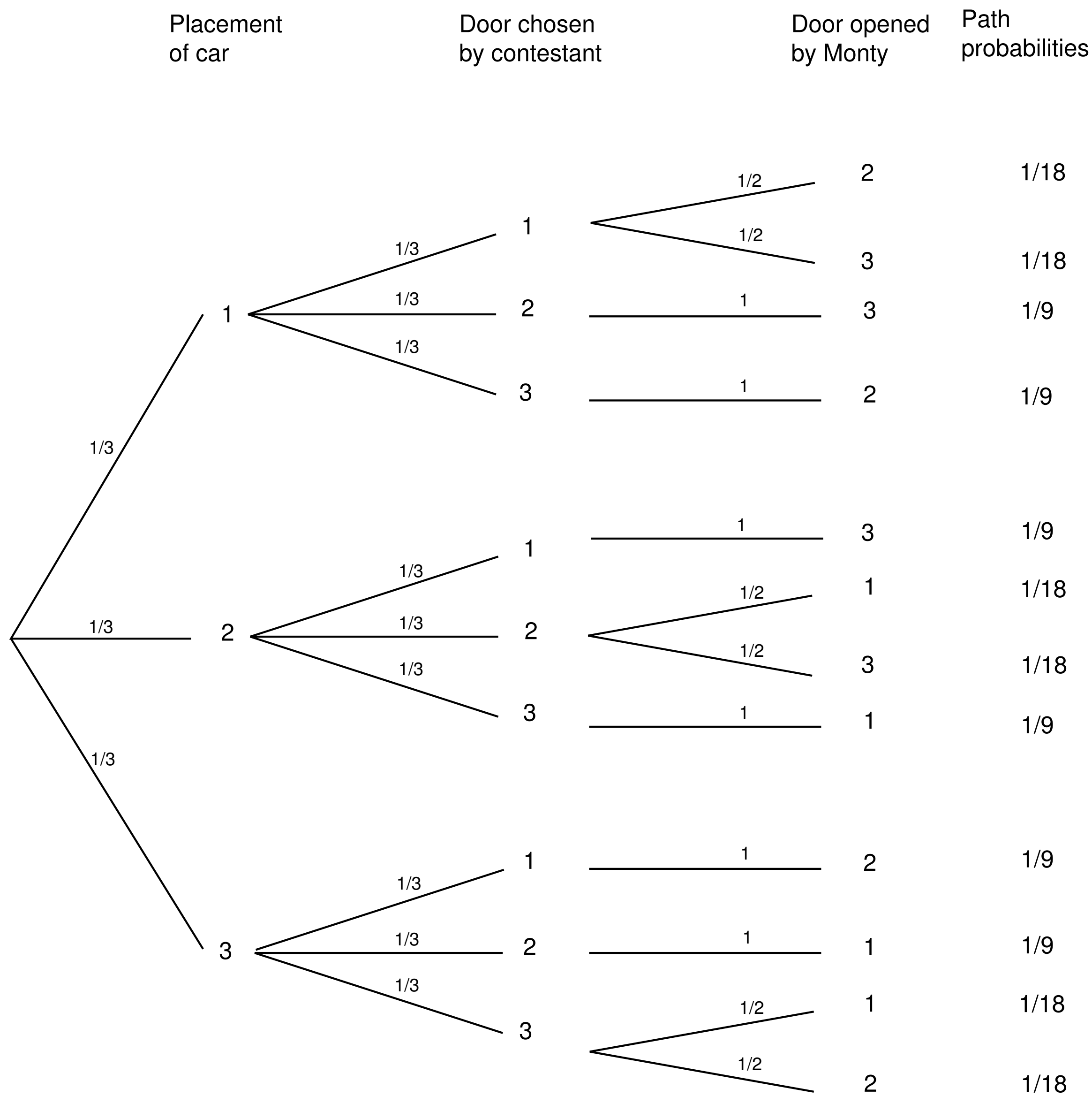
We have just calculated the *inverse probability* that a particular urn was chosen, given the color of the ball. Such an inverse probability is called a *Bayes probability.*


Color of ball
Urn
ω
p (ω)
(start)
9/20
11/20
b
w
4/9
5/9
6/11
5/11
I
II
I
II
ω1
ω3
ω2
ω4
1/5
1/4
3/10
1/4

The Monty Hall problem

Suppose you're on Monty Hall's *Let's Make a Deal!* You are given the choice of three doors, behind one door is a car, the others, goats. You pick a door, say 1, Monty opens another door, say 3, which has a goat. Monty says to you "Do you want to pick door 2?" Is it to your advantage to switch your choice of doors?

Question: What is the conditional probability that you win if you switch, given that you have chosen door 1 and that Monty has chosen door 3.


Placement of car
Door chosen by contestant
Door opened by Monty
Path probabilities
1/3
1/3
1/3
1
2
3
1/3
1/3
1/3
1
2
3
1/2
1/2
1
1
2
3
3
2
1/18
1/18
1/9
1/9
1/3
1/3
1/3
1
2
3
1
1/2
1/2
1
3
1
3
1
1/9
1/18
1/18
1/9
1/3
1/3
1/3
1
2
3
1
1
1/2
1/2
2
1
1
2
1/9
1/9
1/18
1/18

Problem

Assume that E and F are two events with positive probabilities. Show that if $P(E|F) = P(E)$, then $P(F|E) = P(F)$.

Problem

A die is rolled twice. What is the probability that the sum of the faces is greater than 7, given that

1. the first outcome was a 4?

2. the first outcome was greater than 3?

3. the first outcome was a 1?

4. the first outcome was less than 5?