TIPS FOR WRITING PROOFS IN HOMEWORK ASSIGNMENTS

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1. Simple rules

I require my students to follow the rules below when submitting problem sets. While other instructors may be more lenient than I, these rules are likely to help you maximize your homework scores in any class which involves problem sets.

(1) Submit your homework on time, even if it is incomplete. I will not accept late homework.
(2) Begin the solution to each problem on a new page.
(3) Write each solution in claim-proof form, even if the solution is short.
(4) Use full sentences.
(5) Make sure your handwriting or font is large and legible.

You can make your homework even more legible by writing on only one side of a page, but I'll leave this up to you.

2. Claim-proof form

Writing good proofs is not difficult; it just takes practice. The most important element of good proof writing is the use of appropriate full sentences.

Begin each solution with the word "claim", followed by a clear statement of what you wish to prove.

Example 1. *Claim:* The probability of picking a good apple is $\frac{1}{10}$.

Begin the proof with the word "proof", and let your first statement be something which the reader will believe immediately upon reading it. Such statements often start with *Note that*, or *By definition*.

Example 2. *Proof:* Note that

$$\sum_{i=0}^{n}(n-i) = \sum_{i=0}^{n} n - \sum_{i=0}^{n} i$$
$$= n(n+1) - \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)}{2}.$$

Example 3. *Proof:* By the definition of derivative, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

You can use *Recall that* if you think that the reader will require an extra moment to accept your first sentence.

Example 4. *Proof:* Recall that e is equal to $\lim_{n \to \infty}(1 + \frac{1}{n})^n$.

Note that the equations in the above examples are part of full sentences and are punctuated. The words in these sentences help the reader to understand the significance of the equations. Equations which are too cumbersome to be placed in a paragraph are placed in the middle of a line, surrounded by space above and below.

Another appropriate beginning of a proof is the introduction of notation which you plan to use.

Example 5. *Proof:* Let d be the probability that a randomly selected coin was minted in Denver.

A short proof continues with a few manipulations which you should name as you perform them.

Example 6. Differentiating both sides, we have ...

Example 7. Taking the limit as n approaches infinity, we have ...

It ends with statements which follow logically from the previous easy statements. Introduce these conclusions with *Therefore*, *Thus*, or *Combining*.

Example 8. Therefore f must be equal to g.

Example 9. Combining equations (1) and (2), we have $f(0) = 0$.

Longer proofs may involve references to specific theorems, or techniques such as contradiction and induction.

To refer to a theorem which has a name, begin with *By* or *Applying*.

Example 10. By the First Fundamental Theorem of Calculus, we have...

Example 11. Applying the Law of Sines to the expression on the right hand side above, we have...

To refer to a less known theorem, you might need to state it and/or refer to a page number in your textbook.

Example 12. A result concerning limits and exponents (See text, p. 86) states that

$$\lim_{x \to a}[f(x)]^{1/n} = [\lim_{x \to a} f(x)]^{1/n},$$

for any positive integer n. Therefore we may exchange the order of roots and limits in equation (iii) to obtain...

Contradiction is often the easiest way to prove a statement. Begin by supposing that the conclusion of the statement to be proven is false.

Example 13. *Claim:* If the functions $f(x)$ and $g(x)$ are continuous on the interval $[a, c]$, then so is the function $f(x) + g(x)$.

Proof: Suppose that $f(x) + g(x)$ has a discontinuity at $x = b$ for some number b between a and c. Then we have...

State further implications, and arrive at a statement which cannot possibly be true.

Example 14. ...but this implies that $1 = 0$, which is clearly false.

This false statement tells you that your claim must in fact be true.

Induction is another useful technique for writing proofs. Assume that the claim in question is true when it involves a number between 1 and n, and then show that it must be true when this number is equal to $n + 1$.

Example 15. *Claim:* For all positive integers k, we have

$$\sum_{i=0}^{k} i^3 = \left[\frac{k(k+1)}{2}\right]^2.$$

Proof: Assume by induction that the above formula holds whenever k is less than or equal to n. Then we have

$$\begin{aligned}
\sum_{i=0}^{n+1} i^3 &= \sum_{i=0}^{n} i^3 + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4},
\end{aligned}$$

and the identity is true for $k = n + 1$ as well. By induction, the identity holds for all positive integers.

Often easier than *proving* a statement is *disproving* a statement, for the latter requires only a single counterexample.

Example 16. *Claim:* $(fg)'$ is not always equal to $f'g'$.

Proof: The functions

$$f(x) = 1,$$
$$g(x) = \sin(x),$$

satisfy

$$(fg)' = 0 \cdot \sin(x) + 1 \cdot \cos(x) = \cos(x),$$
$$f'g' = 0 \cdot \cos(x) = 0,$$

and $\cos(x)$ is not always equal to zero.

3. Some sample proofs

Here's the solution to a computational problem.

[p.397 #10] *Claim:* The natural length of the spring is 8cm.

Proof. Let ℓ be the natural length of our spring (in m) and let k be its spring constant (in N/m). Using Hooke's Law, we can write two equations for the work done in stretching the spring,

$$(3.1) \qquad \int_{.1-\ell}^{.12-\ell} kx\,dx = 6,$$

$$(3.2) \qquad \int_{.12-\ell}^{.14-\ell} kx\,dx = 10.$$

Integrating (3.1), we have

$$\frac{(.12 - \ell)^2}{2} - \frac{(.1 - \ell)^2}{2} = \frac{6}{k},$$

and solving for ℓ in terms of k, we have

$$\ell = \frac{-.0044}{-.04} - \frac{12}{.04k} = .11 - \frac{300}{k}.$$

Similarly, we integrate (3.2) and solve for ℓ to obtain

$$\ell = \frac{-.0052}{-.04} - \frac{20}{.04k} = .13 - \frac{500}{k}.$$

Equating these two expressions for ℓ, we determine that k is equal to 10000 and therefore that ℓ is equal to .08. □

When writing a paper which includes some proofs and some commentary between the proofs, it is customary to end the proof with a symbol such as the □ above. This probably won't be an issue on your homework assignments.

Here's the solution to a slightly more theoretical problem.

[p.794 #49] *Claim:* The power series expansion of $e^{-x^2} \cos x$ begins with $1 - \frac{3x^2}{2} + \frac{25x^4}{4!}$.

Proof. Using the power series expansion for e^x, we have

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \cdots$$

$$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots .$$

Since the expansion for $\cos x$ is

$$1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots ,$$

we see that the smallest powers of x which will appear in the product are x^0, $x^0 x^2 = x^2 x^0 = x^2$ and $x^0 x^4 = x^2 x^2 = x^4 x^0 = x^4$. Thus we can concentrate just on these terms of the product,

$$(1 - x^2 + \frac{x^4}{2})(1 - \frac{x^2}{2} + \frac{x^4}{4!}) = 1 - \frac{x^2}{2} - x^2 + \frac{x^4}{4!} + \frac{x^4}{2} + \frac{x^4}{2} + \cdots$$

$$= 1 - \frac{3x^2}{2} + \frac{25x^4}{4!} \cdots .$$

□

Finally, here's a mathematician's favorite type of proof: a *one-liner*.

[worksheet 2b] *Claim:* An antiderivative of $\sin^2 x \cos x dx$ is $\frac{\sin^3 x}{3}$.

Proof. Let $u = \sin x$ so that $du = \cos x dx$, and note that an antiderivative of $u^2 du$ is $\frac{u^3}{3}$. □

Don't be afraid to write a short proof. Sometimes it's appropriate.